Resolution of singularities in foliated spaces

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11/2012

Abstract

Let M be an analytic manifold over $\mathbb{R}$ or $\mathbb{C}$, θ an involutive singular distribution and $\mathcal{I}$ a coherent ideal sheaf defined on M. The aim of the work is to study the existence of a resolution of singularities of $\mathcal{I}$ in which nice properties of θ are preserved. This problem has a strong connection with resolution for families of ideal sheafs. We introduce a notion of admissible center, called *θ-admissible center*, which is well-adapted to the singular distribution. We prove two theorems of resolution of singularities under θ-admissible centers: the first when $\mathcal{I}$ is invariant by θ; the second without restrictions on $\mathcal{I}$ but assuming the leaf dimension of θ equals to one.

Contents

1 Introduction

1.1 Results and Main ideas

A *foliated analytic manifold* is a triple (M, θ, E):

- M is a smooth analytic manifold of dimension n over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$);
- E is an ordered collection $E = (E^{(1)}, ..., E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on M such that $\sum_i E^{(i)}$ is a reduced divisor with simple normal crossings;
- θ is an involutive singular distribution defined over M and everywhere tangent to E.

We recall the basic notions of singular distributions (see [3]). Let Der_M denote the sheaf of analytic vector fields over M, i.e. the sheaf of analytic sections of TM. An *involutive singular distribution* is a coherent subsheaf θ of Der_M such that for each $p \in M$, the stalk θ_p is closed under the Lie bracket operation.

Consider the quotient sheaf $Q = Der_M/\theta$. The *singular set* of θ is defined by the closed analytic subset $S = \{p \in M : Q_p \text{ is not a free } \mathcal{O}_p \text{ module}\}$. A distribution θ is called *regular* if $S = \emptyset$. On $M \setminus S$ there exists an unique holomorphic subbundle F of $TM|_{M \setminus S}$ such that θ is the sheaf of holomorphic sections of F. We assume that the dimension of the $\mathbb{K}$ vector space F_p is the same for all $p \in M \setminus S$ (this always holds if M is connected). It will be called the *leaf dimension* of θ and denoted by d. In this case θ is called an involutive *d-singular distribution* and (M, θ, E) a *d-foliated analytic manifold.*

A *coherent set of generators* of θ_p is a set $\{X_1, ..., X_{d_p}\}$ of $d_p \geq d$ vector fields germs with representatives defined in a neighborhood U_p of p such that $\{X_1, ..., X_{d_p}\}.\mathcal{O}_q$ generates θ_q for every $q \in U_p$.

Given any coherent ideal sheaf $\mathcal{I}$ over M, the *support* of $\mathcal{I}$ is the subset:

$$V(\mathcal{I}) := \{p \in M; \mathcal{I}.\mathcal{O}_p \subset m_p\}$$

where m_p is the maximal ideal of $\mathcal{O}_p$.

A blow-up $\sigma : (M', E') \to (M, E)$ is *admissible* if the center $\mathcal{C}$ is a closed and regular sub-manifold that has normal crossings with E. It is admissible *of order one* by $\mathcal{I}$ if it is admissible and $\mathcal{C} \subset V(\mathcal{I})$. When there is no risk of confusion, we abuse notation and simply say that $\mathcal{C}$ is admissible of order one.

The controlled transform over an admissible blow-up of order one $\sigma : (M', E') \to (M, E)$ is the ideal sheaf $\mathcal{I}^c := \mathcal{O}(-F)(\mathcal{I}.\mathcal{O}_{M'})$, where F stands for the exceptional divisor of σ.

A *resolution* of $\mathcal{I}$ is a sequence of blow-ups:

$$(M_r, E_r) \overset{\sigma_r}{\to} ... \overset{\sigma_2}{\to} (M_1, E_1) \overset{\sigma_1}{\to} (M, E)$$

such that each σ_i is an admissible blow-up of order one (by the controlled transform of $\mathcal{I}$) and $\mathcal{I}^c = \mathcal{O}_{M_r}$ (where $\mathcal{I}^c$ is given by the recursive use of the controlled transform). In particular, $\mathcal{I}.\mathcal{O}_{M_r}$ is the ideal sheaf of a SNC divisor on M_r.

We define a natural transform of θ under admissible blow-up: an involutive singular distribution θ' everywhere tangent to E', obtained as a suitable extension of the pull-back of θ from $M \setminus \mathcal{C}$ to $M' \setminus \sigma^{-1}(\mathcal{C})$ (the precise construction is made in section 2.3 and we stress that it is neither the strict nor the total transforms). We note:

$$\sigma : (M', \theta', E') \to (M, \theta, E)$$

Our main objective is to find a resolution algorithm that 'respects' the singular distribution θ. For example if the singular distribution θ is regular, is there a resolution of $\mathcal{I}$ such that the transform of θ is regular?

Unfortunately, it is easy to get examples of ideal sheafs whose resolution necessarily breaks the regularity of a distribution. For example, take $(\mathbb{C}^2, \frac{\partial}{\partial x}, \emptyset)$ and $\mathcal{I} = (x, y)$. The only possible strategy is to blow-up the origin, which breaks regularity of the distribution.

The next best thing is a (locally) *monomial* singular distribution: given a ring R such that $\mathbb{Z} \subset R \subset \mathbb{K}$, a d-singular distribution θ is *R-monomial* at $p \in M$ if there exists a local coordinate system $x = (x_1, ..., x_n)$ and a coherent set of generators $\{X_1, ..., X_d\}$ of θ_p such that:

- Either $X_i = \frac{\partial}{\partial x_i}$;
- Or $X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,j} \in R$;

At this case, we say that $x = (x_1, ..., x_n)$ is *R-monomial coordinate system* and $\{X_1, ..., X_d\}$ is a *R-monomial base* of θ_p. A singular distribution is *R-monomial* if it is R-monomial at every point $p \in M$.

Examples:

- Any regular distribution is a $\mathbb{Z}$-monomial singular distribution;
- We say that a d-singular distribution θ is *R-monomially integrable* at p if there exists a coordinate system $x = (x_1, ..., x_n)$ and monomials (possibly multivalued) functions $\lambda_i = \prod_{j=1}^{n} x_j^{q_{i,j}}$ for $1 \leq i \leq n-d$ with exponents $q_{i,j} \in R$ such that:
  - Each λ_i is a first-integral of all vector-fields in θ_p;
  - And:
  $$\begin{bmatrix} q^{1,1} & ... & q^{1,n} \\ \vdots & \ddots & \vdots \\ q^{d,1} & ... & q^{d,n} \end{bmatrix}$$
  is of maximal rank.

  We say that θ_p is meromorphically (respect. Darboux) monomially integrable if $R = \mathbb{Z}$ (respect. $R = \mathbb{R}$).

We would like to address the following problem:

Main problem: Given an ideal sheaf $\mathcal{I}$ and a R-monomial singular distribution θ, is there a resolution of $\mathcal{I}$:

$$(M_r, \theta_r, E_r) = Bl_{\mathcal{C}_{r-1}}(M_{r-1}, \theta_{r-1}, E_{r-1}) \overset{\sigma_r}{\to} ... \overset{\sigma_2}{\to} (M_1, \theta_1, E_1) \overset{\sigma_1}{\to} (M, \theta, E)$$

such that θ_r is also a R-monomial singular distribution?

A positive answer to this problem would lead to new ways of attacking some well-known problems. We present one interesting possibility:

- (Monomial first integrals): Let θ be an involutive singular totally integrable distribution, i.e. at each point $p \in M$ there exists $(f_1, ..., f_{n-d}) \subset \mathcal{O}_p$ such that $df_1 \wedge ... \wedge df_{n-d}$ is generically of maximal rank and $X(f_i) \equiv 0$ for all $X \in \theta_p$. After a sequence of blow-ups, can we assume that θ is $\mathbb{Z}$-monomially integrable?

Remark 1.1.1. *This monomial first integrals problem has a strong connection with the monomialization of analytic maps (see [7] for the statement of the problem and [7, 4] for some partially positive answers).*

The first difficulty of the main problem is controlling the transform of θ under blowing-ups. We deal with this difficulty restricting the possible centers of blow-up to θ-*admissible centers* (see section 3.1 for the precise definition). Intuitively, a center $\mathcal{C}$ is θ-admissible at a point $p \in \mathcal{C}$ if there exists a local decomposition $\theta_p = \theta_{tr} + \theta_{inv}$ such that:

- θ_{tr} is totally transversal to $\mathcal{C}$;
- θ_{inv} is everywhere tangent to $\mathcal{C}$.

see proposition 3.3.1 for a formalization of this interpretation. This notion is defined for arbitrary singular distributions, but is particularly important for R-monomial singular distributions because of the following result:

Theorem 1.1.2. *Let (M, θ, E) be a R-monomial d-foliated analytic manifold and:*

$$\sigma : (M^{'}, \theta^{'}, E^{'}) \to (M, \theta, E)$$

a θ-admissible blow-up. Then $\theta^{'}$ is R-monomial.

The second difficulty is to find the resolution of an ideal sheaf $\mathcal{I}$ using only θ-admissible centers. This leads to the two main results of this work, that are here presented in their simplest form:

Theorem 1.1.3. *Let (M, θ, E) be a d-foliated analytic manifold, $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero) and $M_0 \subset M$ a relatively compact open set of M. Set $(M_0, \theta_0, E_0) := (M_0, \theta|_{M_0}, E|_{M_0})$ and $\mathcal{I}_0 = \mathcal{I}.\mathcal{O}_{M_0}$. Suppose that $\mathcal{I}_0$ is invariant by θ_0 (i.e. $\theta_0[\mathcal{I}_0] \subset \mathcal{I}_0$), then there exists a resolution of $\mathcal{I}_0$:*

$$\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

such that:

- *Each $\mathcal{C}_i$ is invariant by θ_{i-1} and, in particular, $\mathcal{C}_i$ is a θ_{i-1}-admissible center;*
- *$\sigma : (M_r, E_r) \to (M_0, E_0)$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$ where $\sigma = \sigma_1 \circ ... \circ \sigma_r$;*
- *If θ_0 is R-monomial, then so is θ_r;*

We recall that $\mathcal{I}$ is invariant by θ if for every $p \in M$, $\theta_p[\mathcal{I}_p] \subset \mathcal{I}_p$. An analytic submanifold $\mathcal{C}$ is invariant by θ if its defining ideal sheaf $\mathcal{I}_{\mathcal{C}}$ is invariant by θ (see also section 2.1).

Theorem 1.1.4. *Let (M, θ, E) be a 1-foliated analytic manifold, $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero) and $M_0 \subset M$ a relatively compact open set of M. Set $(M_0, \theta_0, E_0) := (M_0, \theta|_{M_0}, E|_{M_0})$ and $\mathcal{I}_0 = \mathcal{I}.\mathcal{O}_{M_0}$. Then, there exists a resolution of $\mathcal{I}_0$:*

$$\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_1} (M_0, \theta_0, E_0) = (M_0, \theta|_{M_0}, E|_{M_0})$$

such that:

- *Each $\mathcal{C}_i$ is θ_{i-1}-admissible;*
- *$\sigma : (M_r, \theta_r, E_r) \to (M_0, \theta_0, E_0)$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$ where $\sigma = \sigma_1 \circ ... \circ \sigma_r$;*
- *If θ_0 is R-monomial, then so is θ_r;*

In fact theorems 1.1.3 and 1.1.4 are corollaries of the more general theorems 4.2.1 and 5.1.1, where we also prove the functorality of the resolution for a certain kind of morphism.

Remark 1.1.5. *In the case of* 1*-foliated manifolds, θ-admissible blow-ups also preserves canonical singularities (see I.1.2 of* [9] *for the definition) i.e. if θ_0 has only canonical singularities, then θ_r has only canonical singularities (this follows from fact I.2.8 of* [9] *and proposition* 3.3.1 *of this work). More generally, it seems that a similar statement holds for an arbitrary d-foliated manifold.*

Remark 1.1.6. *Although all proves and results of this paper are set in the analytic category, we see no reason why it could not be done in the algebraic category. The proofs should follow the same arguments, but with different local technicalities. We stress that we have not verified all details of this adaptation.*

The main idea to prove theorem 1.1.4 is the introduction of a new invariant. In addition to the classical invariants (see the works of Hironaka [5], Bierstone and Milman [1], Villamayor [15] and Wlodarczyk [18, 19] for example), we focus on the order of tangency between the ideal sheaf and the singular distribution (see section 2.2 for the precise definition). In the particular case where $\theta = TM$, the order of tangency coincides with the usual order of the ideal sheaf.

1.2 Families of ideal sheafs

Classically, a smooth family of ideal sheafs is given by a quadruple $(B, \Lambda, \pi, \mathcal{I})$ where:

- B (the ambient space) and Λ (the parameter space) are two smooth analytic manifolds;
- $\pi : B \to \Lambda$ is a smooth analytic morphism;
- $\mathcal{I}$ is a coherent ideal sheaf over B.

Many works have addressed resolution process for families. In particular, we mention the following:

- Villamayor [16]: The work is set in the algebraic category, and it presents a resolution for embebbed sub-varieties $X \subset B$ such that:

– The parameter space has dimension 1 (more precisely: Λ is a Dedekind scheme);

– X is flat over Λ.

The author uses the notion of quasi-smooth morphisms and Jacobian ideals;

- Villamayor, Encinas and Nobre [17]: The work is set in the algebraic category and it presents a stratification of the parameter space $\Lambda = \cup \Lambda_i$ such that $\mathcal{I}_\lambda$ is "equidesingularizable" over each Λ_i.

In the context of this work, a smooth family gives rise to a foliated analytic manifold (M, θ, E) where θ is the maximal regular distribution such that $(D\pi)\theta = 0$. In particular, we can prove the following:

Theorem 1.2.1. *Let $(B, \Lambda, \pi, \mathcal{I})$ be a smooth family of ideal sheafs such that all fibers are connected. Then, there exists two proper analytic maps $\sigma : B' \to B$ and $\tau : \Lambda' \to \Lambda$ and a smooth map $\pi' : B' \to \Lambda'$ such that:*

- *The following diagram commutes:*

$$\begin{array}{ccc} B' & \xrightarrow{\pi'} & \Lambda' \\ \sigma\downarrow & & \downarrow\tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

- *Given any relatively compact open subset $B_0 \subset B$ (respect. $\Lambda_0 \subset \Lambda$) the analytic map $\sigma|_{\sigma^{-1}B_0}$ (respect. $\tau|_{\tau^{-1}\Lambda_0}$) is a finite sequence of admissible blow-ups of order one;*

- *$(B', \Lambda', \pi', \mathcal{I}')$ is a smooth family, where $\mathcal{I}'$ is given by the direct limit of $\mathcal{I}^c$ over relatively compact open sets of B;*

- *The variety $V(\mathcal{I}')$ does not contain any fiber of π'.*

Proof. In this proof we use all notations and results of the paper.

Consider the two foliated analytic manifolds $(B, \theta, \emptyset)$ and $(\Lambda, 0, \emptyset)$ and let $\mathcal{I}_\#$ be the smaller invariant ideal sheaf containing $\mathcal{I}$. As the fibers of π are connected, there exists an ideal

sheaf $\mathcal{J} \subset \mathcal{O}_\Lambda$ such that $\mathcal{J}.\mathcal{O}_B = \mathcal{I}_\#$ (define the ideal sheaf $\mathcal{J}$ at the stalks using proposition 3.2.2). Then, theorem 4.1.4 guarantees the existence of two proper analytic maps $\sigma : B^{'} \to B$ and $\tau : \Lambda^{'} \to \Lambda$ and a smooth map $\pi^{'} : B^{'} \to \Lambda^{'}$ such that:

- The following diagram commutes:

$$\begin{array}{ccc} B^{'} & \stackrel{\pi^{'}}{\to} & \Lambda^{'} \\ \sigma\downarrow & & \downarrow\tau \\ B & \stackrel{\pi}{\to} & \Lambda \end{array}$$

- Given any relatively compact open subset $B_0 \subset B$ (respect. $\Lambda_0 \subset \Lambda$) the analytic map $\sigma|_{\sigma^{-1}B_0}$ (respect. $\tau|_{\Lambda_0}$) is a finite sequence of admissible blow-ups of order one;
- $(B^{'}, \Lambda^{'}, \pi^{'}, \mathcal{I}^{'})$ is a smooth family, where $\mathcal{I}^{'}$ is given by the direct limit of $\mathcal{I}^c$ over relatively compact open sets of B;
- $\mathcal{I}^c_\# = \mathcal{O}^{'}_M$ and $\mathcal{J}^c = \mathcal{O}_{\Lambda^{'}}$;

Furthermore, the proof of theorem 4.2.2 guarantees that $\mathcal{I}^{'}$ is of type 1 in every point (remember that the searched resolution on 4.2.2 is the resolution of $\mathcal{I}_\#$ by the usual Hironaka's theorem). $\square$

1.3 Example

In this section we give an example in order to show the difficulty of the problem. We work over $(\mathbb{C}^3, \theta, \emptyset)$ and the singular distribution θ is generated by a single vector-field X. In this case a center $\mathcal{C}$ is θ-admissible if:

- Either every orbit of X that intersects $\mathcal{C}$ is contained in $\mathcal{C}$;
- Or every orbit of X that intersects $\mathcal{C}$ is regular and transversal to $\mathcal{C}$.

Now, consider $X = \frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$ and $I = (x, y)$, and let us search a resolution of I by θ-admissible blow-ups. It is clear that, if no restriction existed, we could simply blow-up the line $V(I)$ in order to obtain a resolution.

Notice that X is tangent to the line $V(I)$ at the origin and transversal everywhere else. We can't blow-up the line since it is not θ-admissible and we are forced to blow-up the origin first. At the z-chart we obtain:

$$I^* = (xz, yz) \quad X^* = \frac{1}{z}(z\frac{\partial}{\partial z} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + \frac{\partial}{\partial x}$$
$$I' = (x, y) \qquad X' = z\frac{\partial}{\partial z} + (z - x)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

where (see sections 2.3 and 2.4):

- I^* and X^* are the total transforms;
- I' is the controlled transform
- X' is the adapted analytic-strict transform.

It is clear again that blowing-up $V((x, y))$ would give the desired resolution. But the origin is a singularity of X' and, thus, entirely contained in $V((x, y))$. All other orbits are transversal, which implies that $V((x, y))$ is not θ-admissible. We need to blow-up the origin again. The interesting chart is the z:

$$I'' = (x, y) \quad X'' = z\frac{\partial}{\partial z} + (1 - 2x)\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$$

Now, clearly X'' is transversal to $V((x, y))$ at every point and we can blow-it-up.

The conclusion is that, to resolve I we had to blow-up three times. This shows how θ-admissible blow-ups may complicate the resolution of an ideal I.

2 Main objects

2.1 Generalized k-fitting operations

Let θ be an involutive d-singular distribution over M. The *generalized k-fitting operation* (for $k \leq d$) is a mapping Γ_k that associates to each ideal sheaf $\mathcal{I}$ the ideal sheaf whose stalk at each point $p \in M$ is given by:

$$\Gamma_k(\mathcal{I}).\mathcal{O}_p =< \{det[X_i(f_j)]_{i,j \leq k};\ X_i \in \theta_p, f_j \in \mathcal{I}.\mathcal{O}_p\} >$$

where $< S >$ stands for the ideal generated by the set $S \subset \mathcal{O}_p$.

Remark 2.1.1. *If θ is regular, the operation Γ_k coincides with taking a k-fitting ideal sheaf (see [14]).*

Remark 2.1.2. *If $\theta = Der_M$, the generalized 1-fitting ideal sheaf coincides with the derivative ideal (see chapter 3.7 of [8] for details on derivative ideal sheafs).*

Remark 2.1.3. *If $\mathcal{I}$ is a coherent ideal sheaf, then $\Gamma_k(\mathcal{I})$ is also coherent for every $k \leq d$. This follows from the fact that θ is coherent (thus, locally finitely generated).*

The operation Γ_1 will play an important role in this work. For simplifying the notation, we write $\theta[\mathcal{I}] := \Gamma_1(\mathcal{I})$.

Given a coherent ideal sheaf $\mathcal{I}$, we say that:

- $\mathcal{I}$ is *invariant* by θ if $\theta[\mathcal{I}] \subset \mathcal{I}$;
- $\mathcal{I}$ is *totally transversal* by θ if $\Gamma_d(\mathcal{I}) = \mathcal{O}_M$.

The *differential closure* of $\mathcal{I}$ is the smallest invariant ideal sheaf $\mathcal{I}_\#$ containing $\mathcal{I}$.

At section 4.3 we will give a geometrical interpretation of invariance.

2.2 Chain of Ideal sheafs

A chain of ideal sheafs consists of a sequence $(\mathcal{I}_i)_{i \in \mathbb{N}}$ such that:

- $\mathcal{I}_i$ is an ideal sheaf over $\mathcal{O}_M$;
- $\mathcal{I}_i \subset \mathcal{I}_j$ if $i \leq j$.

The *length* of a chain of ideal sheafs at $p \in M$ is the minimal number $\nu_p \in \mathbb{N}$ such that $\mathcal{I}_i.\mathcal{O}_p = \mathcal{I}_{\nu_p}.\mathcal{O}_p$ for all $i \geq \nu_p$. We distinguish two cases:

- if $\mathcal{I}_{\nu_p}.\mathcal{O}_p = \mathcal{O}_p$, then the chain is said to be of type 1 at p;
- if $\mathcal{I}_{\nu_p}.\mathcal{O}_p \neq \mathcal{O}_p$, then the chain is said to be of type 2 at p.

Fixed a chain of ideal sheaf $(\mathcal{I}_n)$, it is not difficult to see that the functions:

$$\begin{array}{rcl} \nu: M & \to & \mathbb{N} \\ p & \mapsto & \nu_p \end{array} \quad , \quad \begin{array}{rcl} type: M & \to & \{1,2\} \\ p & \mapsto & type_p = \text{ type of } (\mathcal{I}_n) \text{ at } p \end{array}$$

are upper semi-continuous (by the usual order). So, given $U \subset M$ a relatively compact subset, the definition of length and type can be naturally extended to U:

- The length of $(\mathcal{I}_n)$ at U is $\nu_U := max\{\nu_p; p \in U\}$;
- The type of $(\mathcal{I}_n)$ at U is $type_U := max\{type_p; p \in U\}$.

Given an involutive d-singular distribution θ and an ideal sheaf $\mathcal{I}$, the *tangency chain* of $(\theta, \mathcal{I})$ is the following chain of ideal sheafs:

$$\mathcal{C}h(\theta, \mathcal{I}) = \{H(\mathcal{I}, \theta, i); i \in \mathbb{N}\}$$

where the ideal sheafs $H(\mathcal{I}, \theta, i)$ are given by;

$$H(\mathcal{I}, \theta, 0) := \mathcal{I}$$
$$H(\mathcal{I}, \theta, i+1) := H(\mathcal{I}, \theta, i) + \theta[H(\mathcal{I}, \theta, i)]$$

Remark 2.2.1. *The tangency chain does not depend on the set of divisors E.*

Remark 2.2.2. *If θ is an involutive d-singular distribution with $d > 1$, we could have defined non-trivial chains of ideals for every generalized k-fitting operation Γ_k. This won't be necessary in this work.*

At each $p \in M$, the length of this chain is called the *order of tangency* at p, and is denoted by $\nu_p(\mathcal{I}, \theta)$. The type of the chain is denoted by $type_p(\mathcal{I}, \theta)$.

Remark 2.2.3. *Suppose that θ is generated by a vector-field X, non-singular at $p \in V(\mathcal{I})$. The length of the tangency chain at p is called order of tangency because it coincided with the geometrical order of tangency between $V(\mathcal{I})$ and the orbit of X passing through p. Furthermore, if the contact is infinitely tangent then the type of $\mathcal{I}$ at p is 2.*

2.3 The analytic-strict transform

Let $\sigma : M' \to M$ be a blow-up with regular center $\mathcal{C}$ and F the exceptional divisor. Given an involutive d-singular distribution θ over M, we want to define an appropriate transform θ' of θ over M'.

Let $\mathcal{M}_M$ be the sheaf of meromorphic functions over M, and consider $\mathcal{M}_M \otimes_{\mathcal{O}_M} Der_M$ as sheaf of $\mathcal{O}_M$ modules. There exists a natural inclusion of Der_M in $\mathcal{M}_M \otimes_{\mathcal{O}_M} Der_M$:

$$i : \; Der_M \;\; \to \;\; \mathcal{M}_M \otimes_{\mathcal{O}_M} Der_M$$

which, given $U \subset M$ an open subset associates to a vector-field $X \in Der_M(U)$ the element $i(X) = 1 \otimes X \in \mathcal{M}_M(U) \otimes_{\mathcal{O}_M} Der_M(U)$. Notice that this application is injective.

In particular, $i(\theta)$ is a sub-sheaf of $\mathcal{M}_M \otimes_{\mathcal{O}_M} Der_M$. On the other hand, as σ is a morphism, it gives rise to an application on the structural sheafs $\sigma^* : \mathcal{O}_M \to \mathcal{O}_{M'}$ which can be naturally extended to the sheaf of meromorphic function $\sigma^* : \mathcal{M}_M \to \mathcal{M}_{M'}$. Abusing notation, we may finally define:

$$\sigma^* : \mathcal{M}_M \otimes_{\mathcal{O}_M} Der_M \to \mathcal{M}_{M'} \otimes_{\mathcal{O}_{M'}} Der_{M'}$$

as the morphism which, given $U \subset M$ an open subset associates to $(g \otimes X) \in \mathcal{M}_U \otimes Der_U$ the element $\sigma^*(g \otimes X) = (\frac{g}{f} \otimes fX^*)$, where $(f) = \mathcal{O}(F).\mathcal{O}_{\sigma^{-1}(U)}$ and X^* is the pull-back of the derivation (where $X^*(\sigma^* f) = \sigma^* X(f)$).

The necessity to consider derivations on meromorphic functions is illustrated by the following example:

Example: $M = \mathbb{C}^2$, $X = \frac{\partial}{\partial x}$ and let $(x, y) = I$ be the center of blow-up. Then:

- In the x-chart $X^* = \frac{1}{x}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$;
- In the y-chart $X^* = \frac{1}{y}\frac{\partial}{\partial y}$.

In particular, even though θ is analytic, we can't guarantee that $\sigma^* i(\theta)$ is analytic.

Remark 2.3.1. *One can see that σ^* is well-defined by explicit calculation. Indeed, the blow-up of an analytic vector-field has at most poles (as in the previous example). We leave the details to the reader.*

The image $\sigma^*(i(\theta))$ is a a coherent sub-sheaf of $\mathcal{M}_{M'} \otimes_{\mathcal{O}_{M'}} Der_{M'}$ (apart from completing it on the new open sets of $M^{'}$). Furthermore, it respects two important properties:

- i) $[\sigma^* i(X), \sigma^* i(Y)] = \sigma^* i([X,Y])$ for vector-fields X and Y contained in θ_U for some open set $U \subset M$;
- ii) $\sigma^* i(X)(\sigma^* h) = \sigma^*(X(h))$ for every vector-field $X \in \theta_U$ and $h \in \mathcal{O}_U$ for some open set $U \subset M$.

The sheaf $\sigma^*(i(\theta))$ is called the *total transform* of θ. We will abuse notation and denote it by $\sigma^*\theta$ or simply θ^*.

Consider the subset θ^a of $Der_{M'}$ given by the pre-image of θ^*, i.e. for each open set $U \subset M^{'}$, define $\theta^a_U = \{X \in Der_U;\ i(X) \in \theta^*_U\}$. Then:

Lemma 2.3.2. *The sub-set θ^a has a natural structure of an involutive d-singular distribution. Moreover, let $q \in M^{'}$, $p = \sigma(q)$ and $\{X_1, ..., X_{d_p}\}$ be a coherent set of generators of θ_p. Then θ^a_q has a coherent set of generators $\{Y_i, Z_j, W_k\}$ with $i = 1, ..., r$, $j = 1, ..., s$ $(r + s = d_p)$ and $k = 1, ..., t$, where:*

- $Y_i = (\mathcal{O}(F) X^*_i).\mathcal{O}_q$ *whenever* $X^*_i.\mathcal{O}_q$ *is not analytic;*
- $Z_j = X^*_j.\mathcal{O}_q$ *whenever* $X^*_j.\mathcal{O}_q$ *is analytic;*
- $W_k = \mathcal{O}(-F) \sum \gamma_{i,k} Y_i$ *for some* $\gamma_k \in \mathcal{O}^r_U$ *such that* $W_k \notin < Y_i, Z_j >$.

Proof.
- Pre-sheaf structure: Notice that θ^a_U is a module over $\mathcal{O}_{M'}$. Indeed, if $X, Y \in \theta^a_U$ and $f \in \mathcal{O}_M$, then, it is clear that $i(X + Y) = i(X) + i(Y) \in \theta^*$ and that $i(fX) = i(f)i(X) \in \theta^*$. For each inclusion of open sets $U \subset V$, consider $r_{U,V}$ as the restriction of the derivations in θ^a_U to θ^a_V. It is clearly well-defined because θ^* is a sheaf.

- Sheaf: The local identity follows immediately from the fact that σ^* is a sheaf and i is injective for each open set $U \subset M'$. Gluing: let (U_i) be an open cover of an open set U and $X_i \in \theta^a_{U_i}$ vector-fields that coincide at each $U_i \cap U_j$. This implies that $i(X_i)$ respect the same property, and, as θ^* is a sheaf, there exists $Y \in \theta^*_U$ such that $r_{U,U_i} Y = i(X_i)$. We claim that there exists $X \in \theta^a_U$ such that $i(X) = Y$, which ends the proof. Indeed, suppose by absurd that there does not exist. This would imply that Y is not analytic at some point $p \in U$. As p must be contained in some U_i this contradicts the fact that X_i is analytic for all i, which is an absurd. Thus, θ^a is a sheaf.

- Coherence: If $q \notin F$ the result is clear because σ is a local isomorphism and $i : \theta^a_q \to \theta^*_q$ is an isomorphism. So, take $q \in F$ and $p = \sigma(q)$. If $\{X_1, ..., X_{d_p}\}$ is a coherent set of generators of θ_p, then it is clear that $\theta^*_q =< \sigma^*(i(X_1)), ..., \sigma^*(i(X_{d_p})) > .\mathcal{O}_q =< (\frac{1}{f} \otimes f X^*_1), ..., (\frac{1}{f} \otimes f X^*_{d_p})) > .\mathcal{O}_q$.

  So take U a sufficiently small neighborhood of q and $(x, y) = (x, y_1, ..., y_{n-1})$ a coordinate system such that $f = x$ and $\theta^a_U =< (\frac{1}{x} \otimes x X^*_1), ..., (\frac{1}{x} \otimes x X^*_{d_p}) > .\mathcal{O}_U$. Notice that whenever $X^*_i.\mathcal{O}_U$ is an analytic vector-field, $(1 \otimes X^*_i.\mathcal{O}_U) = (\frac{1}{x} \otimes x X^*_i.\mathcal{O}_U)$. Reorganizing the set of generators, we can suppose that $\sigma^*_U =< (\frac{1}{x} \otimes Y_1), ..., (\frac{1}{x} \otimes Y_r), (1 \otimes Z_1), ..., (1 \otimes Z_s) >$ where $r + s = d_p$, $Y_i = x X^*_i.\mathcal{O}_U$ (such that $Y_i(0, y) \not\equiv 0$) and $Z_i = X^*_i.\mathcal{O}_U$.

  Let $\mathcal{R}$ be the sub-module of relations of $\{Y_i|_{x=0}\}$, i.e. the r-tuples $(f_1, ..., f_r) \in \mathcal{O}^r_U$ such that $(\sum_{i=1}^r f_i Y_i)|_{x=0} \equiv 0$. It is easy to see that this is the same sub-module of relations of $\{Y_i(x)|_{x=0}, Y_i(y_j)|_{x=0}\}_{i \leq r, j \leq n-1}$. Thus, by the Oka's theorem (see theorem 6.4.1 of [6]), $\mathcal{R}$ is finitely generated: $\mathcal{R} = (F_1, ..., F_t)$ where $F_i = (f_{1,i}, ..., f_{r,i})$.

  In particular, for every $j \leq t$, $\sum f_{i,j} Y_i$ is divisible by x. So, for each F_j, we have that:

$$\sum_{i=1}^r (\frac{f_{i,j}}{x} \otimes Y_i) = (\frac{1}{x} \otimes \sum_{i=1}^r f_{i,j} Y_i) = (1 \otimes W_j)$$

  We claim that $\{Y_i, Z_j, W_k\}_{i \leq r, j \leq s, k \leq t}$ generates θ^a_U, which implies coherence. For this,

take $X \in \theta^a_U$. We only need to show that $i(X) \in \{i(Y_i), i(Z_j), i(W_k)\}_{i\leq r, j\leq s, k\leq t}$. We know there exists $\alpha \in \mathcal{O}^r_U$ and $\beta \in \mathcal{O}^s_U$ such that:

$$i(X) = (1 \otimes X) = \sum \alpha_i(\frac{1}{x} \otimes Y_i) + \sum \beta_j(1 \otimes Z_j)$$

Now, $\alpha_i = x\widetilde{\alpha}(x,y)_i + \bar{\alpha}_i(y)$ and thus:

$$i(X) = \sum \widetilde{\alpha}_i(x,y)(1 \otimes Y_i) + \sum \beta_j(1 \otimes Z_j) + \sum \bar{\alpha}_i(y)(\frac{1}{x} \otimes Y_j)$$

It is clear that $\sum \bar{\alpha}_i(y)Y_i$ is divisible by x. This implies that $(\bar{\alpha}_i) \subset \mathcal{R}$. So, there exists $\gamma \in \mathcal{O}^t_U$ such that $(\bar{\alpha}) = \sum \gamma_k F_k$. This finally implies that:

$$i(X) = \sum \widetilde{\alpha}_i(x,y)(1 \otimes Y_i) + \sum \beta_j(1 \otimes Z_j) + \sum \gamma_k(1 \otimes W_k)$$

- Closed by Lie Brackets: For any $q \in M'$, let X, $Y \in \theta'_q$. Then $i(X)$, $i(Y) \in \theta^*_q$. As θ^*_q is closed by Lie brackets, $[i(X), i(Y)] \in \theta^*_q$. On the other hand, the Lie bracket of analytic derivations is an analytic derivation. This implies that $[X,Y] \in \theta^a_q$.
- Leaf dimension: As σ is bimeromorphic, and $i : \theta^a \to \theta^*$ is an isomorphism for every point outside the exceptional divisor, θ^a has also leaf dimension d.

□

The involutive d-singular distribution θ^a is called *analytic-strict* transform of θ.

Consider the sub-sheaf $Der_{M'}(-logF) \subset Der_{M'}$ composed all the derivations that leave F invariant. We leave to the reader to show that $Der_{M'}(-logF)$ is an involutive n-singular distribution. The *adapted analytic strict* transform of θ is defined as $\theta' = \theta^a \cap Der_{M'}(-logF)$: it is a coherent and closed by Lie brackets sub-sheaf of $Der_{M'}$ (the coherence is a consequence of Oka's Theorem) and, thus, an involutive d-singular distribution.

For avoiding carrying the morphism $i : Der_M \to \mathcal{M}_M \otimes_{\mathcal{O}_M} Der_M$ through the work, whenever ω is a sub-sheaf of $\mathcal{M}_M \otimes_{\mathcal{O}_M} Der_M$ isomorphic to its image i.e. ω is isomorphic to $i(\omega)$, we will abuse notation and denote $i(\omega)$ by ω. In particular, if θ^* is analytic, then we will also denote by θ^* the sub-sheaf $i^{-1}(\theta^*)$.

2.4 Transforms - Notations

Let $\sigma : M' \to M$ be a admissible blow-up by a regular closed irreducible sub-manifold center $\mathcal{C}$ and F the exceptional divisor. We recall the standard notions of total, controlled and strict transforms of sets and ideal sheafs:

- Let $S \subset M$ be any set. We define two transforms:
  - $S^* = \sigma^* S = \sigma^{-1} S$ is the total transform of S;
  - $S^s := \sigma^s S = \overline{\sigma^{-1}(S \setminus \mathcal{C})}$ is the strict transform of S (where $\overline{S}$ stands for the topological closure of S).
- Let $\mathcal{I}$ be a coherent ideal sheaf over M. We define two transforms:
  - $\mathcal{I}^* := \sigma^* \mathcal{I} = \mathcal{I}.\mathcal{O}_{M'}$ is the total transform of $\mathcal{I}$;
  - Suppose that $\mathcal{C} \subset V(\mathcal{I})$, then $\mathcal{I}^c := \mathcal{I}^*.\mathcal{O}(-F)$ is the controlled transform of $\mathcal{I}$.

Remark 2.4.1. *In the language of marked ideals (see e.g. chapter 3.5 of [8]) the controlled transform of an ideal sheaf corresponds precisely to the weak or birational transform of the marked ideal sheaf* $(\mathcal{I}, 1)$ *(see of definition 3.60 of [8]).*

Given a d-foliated analytic manifold (M, θ, E), we say that $\mathcal{C}$ is an *admissible* center if $\mathcal{C}$ is a closed regular sub-manifold which has SNC with E. It is admissible *of order one* by $\mathcal{I}$ if $\mathcal{C}$ is admissible and $\mathcal{C} \subset V(\mathcal{I})$. We say that a blow-up $\sigma : M' \to M$ is *admissible* if the center $\mathcal{C}$ is admissible. At this case, we define the transform (M', θ', E') where:

- $M' = Bl_{\mathcal{C}}(M)$;
- θ' is the adapted analytic-strict transform;
- $E' = ((E^{(1)})^s, ..., (E^{(l)})^s, F)$.

Remark 2.4.2. *The divisor E' has SNC because we assume that $\mathcal{C}$ has SNC with E. We leave to the reader the verification that E' is invariant by θ'.*

A *sequence of admissible blow-ups* is a sequence $\sigma = (\sigma_r, ..., \sigma_1)$, where:

$$(M_r, \theta_r, E_r) \overset{\sigma_r}{\to} ... \overset{\sigma_1}{\to} (M_0, \theta_0, E_0)$$

and for $1 \leq i \leq r$, $\sigma_i : (M_{i+1}, \theta_{i+1}, E_{i+1}) \to (M_i, \theta_i, E_i)$ is an admissible blow-up with center $\mathcal{C}_i$. At this case, given $p_i \in M_i$ and $\mathcal{I}$ a coherent ideal sheaf over M, we establish the following notations:

- The exceptional divisor of σ_i is denoted by $F_{i+1} \subset M_{i+1}$;
- $(i\sigma) := (\sigma_{i+1} \circ ... \circ \sigma_r)$;
- $(\sigma i) := (\sigma_1 \circ ... \circ \sigma_i)$;
- $\theta_{i,p_i} := \theta_i.\mathcal{O}_{p_i}$;
- $\mathcal{I}^{*,i} := \mathcal{I}.\mathcal{O}_{M_i}$;
- If all blow-ups are admissible of order one, $\mathcal{I}^{c,i} := \sigma_i^c(\mathcal{I}^{c,i-1})$ is defined recursively.

3 θ-Admissible Blow-ups

3.1 Definition and main Result

Let (M, θ, E) be a d-foliated analytic manifold and $\mathcal{C}$ an analytic subset of M. Let $\mathcal{I}_\mathcal{C}$ be the radical ideal sheaf that generates $\mathcal{C}$, i.e. $V(\mathcal{I}_\mathcal{C}) = \mathcal{C}$ and $\sqrt{\mathcal{I}_\mathcal{C}} = \mathcal{I}_\mathcal{C}$. We say that $\mathcal{C}$ is a θ-*admissible* center if:

- $\mathcal{C}$ is a regular irreducible closed sub-manifold;
- $\mathcal{C}$ has SNC with E;
- There exists $0 \leq n_0 \leq d$ such that $\Gamma_k(\mathcal{I}_\mathcal{C}) = \mathcal{O}_M$ for all $k \leq n_0$ and $\Gamma_k(\mathcal{I}_\mathcal{C}) \subset \mathcal{I}_\mathcal{C}$ otherwise (where $\Gamma_k(\mathcal{I}_\mathcal{C})$ denotes the generalized k-fitting ideal sheaf associated to $\mathcal{I}_\mathcal{C}$).

In particular, if a center $\mathcal{C}$ is admissible and invariant (or totally transversal) by θ, it is θ-admissible. But these centers don't represent all possible θ-admissible centers as shown in

the following example:

Example: $M = \mathbb{C}^3$ and θ generated by $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. Then:

- $\mathcal{C} = \{x = 0\}$ is θ-admissible but it is neither invariant nor totally transversal. Indeed, $\Gamma_1(\mathcal{I}_{\mathcal{C}}) = \mathcal{O}_M$ and $\Gamma_2(\mathcal{I}_{\mathcal{C}}) \subset \mathcal{I}_{\mathcal{C}}$;
- $\mathcal{C} = \{x^2 = z\}$ is not θ-admissible. Indeed $\Gamma_1(\mathcal{I}_{\mathcal{C}}) = (x, z)$.

A blow-up $\sigma : \widetilde{M} \to M$ is *θ-admissible* if the center $\mathcal{C}$ is θ-admissible. A *sequence of θ-admissible blow-ups* is a sequence $\sigma = (\sigma_r, ..., \sigma_1)$, where:

$$(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

and for $1 \leq i \leq r$, $\sigma_i : (M_{i+1}, \theta_{i+1}, E_{i+1}) \to (M_i, \theta_i, E_i)$ is a θ_i-admissible blow-up. The following theorem justifies the use of such centers:

Theorem (1.1.2): Let (M, θ, E) be a R-monomial d-foliated analytic manifold and:

$$\sigma : (M^{'}, \theta^{'}, E^{'}) \to (M, \theta, E)$$

a θ-admissible blow-up. Then $\theta^{'}$ is R-monomial.

In the process of proving the theorem we give a geometrical interpretation of θ-admissible centers (see remark 3.3.2). For making the proof clearer, we divide it in several propositions and lemmas presented in three subsections. The two first subsections prove the existence of "good" coordinate systems. The proof of the theorem is given in subsection 3.4.

3.2 R-monomial coordinates for an invariant θ-admissible center

At this subsection, we prove the following proposition:

Proposition 3.2.1. *Let (M, θ, E) be a R-monomial d-foliated analytic manifold and $\mathcal{C}$ an invariant θ-admissible center (i.e. $\theta[\mathcal{I}_{\mathcal{C}}] \subset \mathcal{I}_{\mathcal{C}}$). Then, at each point $p \in \mathcal{C}$, there exists a R-monomial coordinate system $x = (x_1, ..., x_n)$ such that $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p = (x_1, ..., x_t)$.*

In what follows, $\mathcal{C}$ is always an invariant θ-admissible center and, given $p \in M$, we denote by $I_{\mathcal{C}} = \mathcal{I}_{\mathcal{C}}.\mathcal{O}_p$ if there is no risk of confusion on the point p.

The fundamental step for proving proposition 3.2.1 is the following result:

Lemma 3.2.2. *Let (M, θ, E) be a R-monomial d-foliated analytic manifold and $\mathcal{I}$ an invariant regular coherent ideal sheaf. Given a point $p \in M$ and a R-monomial coordinate system $x = (x_1, ..., x_n)$ with a R-monomial base $\{X_1, ..., X_d\}$, there exists a set of generators $\{f_1, ..., f_t\}$ of $I := \mathcal{I}.\mathcal{O}_p$ such that:*

- *$X_i(f_j) \equiv 0$ if X_i is regular;*
- *$X_i(f_j) = K_{i,j} f_j$ for some $K_{i,j} \in R$, if X_i is singular.*

Let us see how this result proves proposition 3.2.1:

Proof. (of proposition 3.2.1) Take $p \in \mathcal{C}$. Our proof is by induction on the pair (d, n), where d is the minimal set of generators for θ_p (that coincides with the leaf dimension of θ when θ is R-monomial) and n is the ring dimension of $\mathcal{O}_p$ (that coincides with the dimension of M because M is everywhere regular).

Notice that for $d = 0$ or $n = 1$ the result is trivial (if $n = 1$, the support of the ideal is a point). By induction, suppose that for all $(d', n') < (d, n)$ (by the lexicografical order) there is always a R-monomial coordinate system where $I_{\mathcal{C}} = (x_1, ..., x_t)$. We prove it to (d, n).

Fix a R-monomial coordinate system and $\{X_1, ..., X_d\}$ a R-monomial base. By lemma 3.2.2, there exists a set of generators $\{f_1, ..., f_t\}$ of $I_{\mathcal{C}}$ such that:

- $X_i(f_j) \equiv 0$ if X_i is regular;
- $X_i(f_j) = K_{i,j} f_j$ for some $K_{i,j} \in R$, if X_i is singular.

We have two cases to consider:

- Case I: There exists a regular vector-field on the R-monomial base $\{X_1, ..., X_d\}$. Without loss of generality, suppose $X_1 = \frac{\partial}{\partial x_1}$ and that $X_j(x_1) = 0$ for all $j \neq 1$. Then, as $X_1(f_i) \equiv 0$ for all i, the set of generators is independent of x_1.

  Consider the quotient $\mathcal{O}_p/(x_1)$. It is a regular space of dimension $n-1$. The image of the distribution is a R-monomial involutive $d-1$-singular distribution given by the image of X_i, for $i > 1$. By induction, there exists a change of coordinates such that $\bar{I}_{\mathcal{C}} = (\bar{x}_2, ..., \bar{x}_{t+1})$. Doing the equivalent change of coordinates in $\mathcal{O}_p$, as the change is invariant by x_1, we get $I_{\mathcal{C}} = (x_2, ..., x_{t+1})$.

- Case II: All vector-fields of the R-monomial base $\{X_1, ..., X_d\}$ are singular:

$$X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$$

  As $I_{\mathcal{C}}$ is regular, we can suppose that f_1 is regular and, without loss of generality, that $\frac{\partial}{\partial x_1} f_1(p) \neq 0$. Take the change of coordinates $\bar{x}_1 = f_1$ and $\bar{x}_i = x_i$ otherwise. In the new coordinates, we get:

$$\bar{X}_i = \sum_{j=2}^{n} \alpha_{i,j} \bar{x}_j \frac{\partial}{\partial \bar{x}_j} + K_{1,i} \bar{x}_1 \frac{\partial}{\partial \bar{x}_1}$$

  because $v_i(f_1) = K_{1,i} f_1$ for $K_{1,j} \in R$. Notice that $[\bar{X}_1, ..., \bar{X}_d]$ is also a R-monomial base. We drop the bars of this coordinate system.

  Consider the quotient $\mathcal{O}_{M,p}/(x_1)$. we get a space of dimension $n-1$ with $\bar{\theta}_p = \theta/(x_1)$ a singular distribution satisfying the following conditions:

  - Either $\bar{\theta}_p$ is a R-monomial singular distribution of dimension d;
  - Or a $\bar{\theta}_p$ is a R-monomial singular distribution of dimension $d-1$ and we can assume $X_1 = x_1 \frac{\partial}{\partial x_1}$ (because $\bar{X}_1 \in < \bar{X}_2, ..., \bar{X}_d >$).

  Either way, by induction, there exists a R-monomial coordinate system such that $\bar{I}_{\mathcal{C}} = (\bar{x}_2, ..., \bar{x}_{t+1})$. Doing the equivalent change of coordinates in $\mathcal{O}_p$, as the change is invariant by x_1, we conclude the result.

$\square$

For proving lemma 3.2.2, we will need some preliminary definitions:

- Fixed a coordinate system $x = (x_1, ..., x_n)$, let $\widehat{\mathcal{O}}_p$ denote the completion of $\mathcal{O}_p$. We introduce the topology of simple convergence, defined by a countably many seminorms:

$$f = \sum a_\alpha x^\alpha \to |a_\alpha|$$

  Thus $f_i \to f$ means that the coeficients of x^α in f_i converges to the coefficient of x^α in f;

- Fixed a coordinate system $x = (x_1, ..., x_n)$, and given $\alpha \in \mathbb{N}^n$, we denote by δ^α the derivation $\frac{\partial^{\alpha_1}}{\partial x_1} ... \frac{\partial^{\alpha_n}}{\partial x_n}$. Given two functions $f, g \in \mathcal{O}_p$ we say that g *is contained in the Taylor expansion of* f if, for all α, either $\delta^\alpha g(p) = \delta^\alpha f(p)$ or $\delta^\alpha g(p) = 0$.

We also recall the following result (see section 6.3 and Theorems 6.3.4 and 6.3.5 of [6]):

Proposition 3.2.3. *Let I be an ideal of $\mathcal{O}_p$ and $(f_n)_{n \in \mathbb{N}} \in I$ be a sequence of analytic germs such that (f_n) converges simply to an analytic function germ f. Then, $f \in I$.*

We start the proof of lemma 3.2.2 supposing that θ has leaf dimension 1. In the next lemma, the coordinate system $x = (x, y_1, ..., y_{n-1})$ is fixed:

Lemma 3.2.4. *In the notation of lemma 3.2.2, if θ_p has leaf dimension 1 and $\theta_p =< X = \frac{\partial}{\partial x} >$, then there exists a set of generators $(h_1, ..., h_t)$ of I such that $X(h_i) \equiv 0$. Moreover, if $(f_1, ..., f_r)$ is any set of generators of I, we can choose $(h_1, ..., h_t)$ such that each h_j is contained in the Taylor expansion of a f_i.*

Proof. Take $(f_1, ..., f_r)$ any set of generators of I and set $f = f_1$. Consider its Taylor expansion in x:

$$f = \sum_{i=0}^{\infty} h_i(y) x^i$$

As I is invariant by X, we have that $(f)_\# \subset I$ (we recall that $(f)_\#$ is the differential closure of the ideal (f)). We claim that $(h_i(y))_{i \in \mathbb{N}} = (f)_\#$.

Indeed, let us prove that $h_0(y) \in (f)_\#$ (the others are analogous). We define $g_0 = f$ and we define recursively the expressions:

$$g_{i+1} := g_i - i^{-1} x X(g_i)$$

It is easy to see that:

$$g_i = h_0(y) + \sum_{j=i}^{\infty} \beta_{i,j} h_j(y) x^j$$

for some $\beta_{i,j} \in \mathbb{C}$. It is clear that $(g_n) \subset I$ converges simply to $h_0(y)$. By proposition 3.2.3, this implies that $h_0(y) \in (f)_\# \subset I$. Repeating the process for every $i \in \mathbb{N}$, we conclude that $h_i(y) \in I$ for all i. Thus $(h_i(y))_{i\in\mathbb{N}} \subset (f)_\#$.

Using again proposition 3.2.3, it is clear that $(h_i(y))_{i\in\mathbb{N}} \supset (f)_\#$. Moreover, as the ring is noetherian, we have that $(h_i(y))_{i \leq N} = (f)_\#$ for some $N \in \mathbb{N}$. Doing this for all the generators of I, we get the desired result. □

In the next lemma, the R-monomial coordinate system $x = (x_1, ..., x_n)$ is fixed:

Lemma 3.2.5. *In the notation of proposition* 3.2.2*, if θ_p has leaf dimension* 1 *and $\theta_p =< X >$ where X is a singular R-monomial vector-field, then there exists a set of generators $(h_1, ..., h_t)$ of I such that $X(h_i) = K_i h_i$, for $K_i \in R$. Moreover, if $(f_1, ..., f_r)$ is any set of generators of I, we can choose $(h_1, ..., h_t)$ such that each h_j is contained in the Taylor expansion of a f_i.*

Proof. Let $(f_1, ..., f_r)$ be a set of generators of I and set $f = f_1$. As the coordinate system is R-monomial we have that $X = \sum_{i=1}^{n} K_i x_i \frac{\partial}{\partial x_i}$ for $K_i \in R$. Taking any monomial $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ we get:

$$X(x^\alpha) = \sum_{i=1}^{n} K_i \alpha_i x^\alpha = K_\alpha x^\alpha$$

For some $K_\alpha \in R$ (because $\alpha_i \in \mathbb{Z}$ and $K_i \in R$). As the number of different monomials is countable, there exists a countable set $R' \subset R$ such that $K_\alpha \in R'$, for all $\alpha \in \mathbb{Z}^n$. This allow us to rewrite the Taylor expansion of $f = f_1$ in the following form:

$$f(x) = \sum_{i\in\mathbb{N}} h_i(x)$$

with $h_i(x)$ such that $Xh_i(x) = K_i h_i(x)$, $K_i \in R^{'}$ and $K_i \neq K_j$ whenever $i \neq j$. Moreover, as there exists a representative of f convergent in a open neighborhood of p (thus absolutely convergent), $h_i(x) \in \mathcal{O}_p$. We claim that $(h_i(x))_{i\in\mathbb{N}} = (f)_{\#}$. Indeed, we show that $h_0 \in (f)_{\#}$ (the others are analogous). Define $g_0 = f$ and:

$$g_1 := \frac{1}{K_0 - K_1}(K_1 f - X(f)) = \frac{1}{K_0 - K_1}[\sum_{i\in\mathbb{N}} K_i h_i(x) - K_1 \sum_{i\in\mathbb{N}} K_i h_i(x)] = h_0 + \sum_{i\geq 2} \beta_{i,1} h_i \in (f)_{\#}$$

where $\beta_{i,1} = \frac{K_i - K_1}{K_0 - K_1}$. We define recursively:

$$g_n = \frac{1}{K_0 - K_n}(K_n g_{n-1} - X(g_{n-1})) = h_0 + \sum_{i\geq n+1} \beta_{i,n} h_i \in (f)_{\#}$$

for non-zero constants $\beta_{i,n}$. It is clear that $(g_n) \subset I$ converges simply to $h_0(x)$. By the proposition 3.2.3, this implies that $h_0(x) \in (f)_{\#}$. Repeating the process for every $i \in \mathbb{N}$, we conclude that $(h_i(y)) \subset (f)_{\#}$ for all i.

Using again proposition 3.2.3, it is clear that $(h_i(y))_{i\in\mathbb{N}} \supset (f)_{\#}$. Moreover, as the ring is noetherian, we have that $(h_i(y))_{i\leq N} = (f)_{\#}$ for some $N \in \mathbb{N}$. Doing this for all f_i in the set of generators of I, we conclude the result. $\square$

We are ready to prove lemma 3.2.2:

Proof. (lemma 3.2.2): We prove the result by induction on the leaf dimension of θ. Fix a R-monomial coordinate system $x = (x_1, ..., x_n)$ and a R-monomial base $\{X_1, ..., X_d\}$. Let $(f_1, ..., f_t)$ be a set of generators of I and assume by induction that the lemma is true for $d^{'} < d$.

By the induction hypotheses, we can assume without loss of generality that:

- $X_i(f_j) \equiv 0$ if X_i is regular;
- $X_i(f_j) = K_{i,j} f_j$ for some $K_{i,j} \in R$, if X_i is singular.

for all $i < d$. Now, by lemma 3.2.4 or 3.2.5 there exists another set of generators $(h_1, ..., h_l)$ such that:

- Either $X_d(h_j) \equiv 0$ if X_d is regular;

- $X_d(h_j) = K_{d,j} h_j$ for some $K_{i,j} \in R$, if X_d is singular.

Furthermore, as each h_i is a part of the Taylor expansion of some f_j, we have that:

- $X_i(h_j) \equiv 0$ if X_i is regular;
- $X_i(h_j) = K_{i,j} h_j$ for some $K_{i,j} \in R$, if X_i is singular.

for all $i \leq d$. □

3.3 Local coordinates for a θ-admissible center

In what follows, $\mathcal{C}$ is always a θ-admissible center. Given $p \in M$, we denote $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p$ by $I_{\mathcal{C}}$ if there is no risk of confusion on the point p. We prove the following proposition:

Proposition 3.3.1. *Let (M, θ, E) be a d-foliated analytic manifold and $\mathcal{C}$ a θ-admissible center. Then, at each point $p \in \mathcal{C}$, there exists a coherent generator set $< Y_i, Z_j >= \theta_p$ such that:*

- *$I_{\mathcal{C}}$ is totally transversal to $\{Y_i\}$;*
- *$I_{\mathcal{C}}$ is invariant by $\{Z_j\}$;*
- *There exists a coordinate system $x = (x_1, ..., x_n)$ such that: $I_{\mathcal{C}} = (x_1, ..., x_t)$, $Y_i = \frac{\partial}{\partial x_i}$ and $Z_j(x_i) = 0$;*
- *If θ is R-monomial, then there exists a R-monomial coordinate system $x = (x_1, ..., x_n)$ such that $< Y_i, Z_j >$ is a R-monomial base. Moreover, this coordinate system is the same as the one of the third statement.*

Proof. We prove the lemma for θ a R-monomial singular distribution. In the general case, one only have to prove the first three statements and can be more clumsy with the coordinate systems. As we will see, in the case of a R-monomial singular distribution, not only this hypotheses don't help, as it obligates us to be careful with the coordinate changes.

Fix $p \in \mathcal{C}$ and take a R-monomial coordinate system $x = (x_1, ..., x_n)$ and a R-monomial

base $\{X_1, ..., X_d\}$. If $\mathcal{C}$ is invariant by θ, the proposition trivially follows from proposition 3.2.2. So, suppose that $\mathcal{C}$ is not invariant by θ. There exists a maximal $n_0 > 0$ such that $\Gamma_{n_0}(\mathcal{I}_{\mathcal{C}}) = \mathcal{O}_M$. This implies that there exists $(f_1, ..., f_{n_0}) \subset I_{\mathcal{C}}$ such that:

$$\left\| \begin{matrix} X_1(f_1) & ... & X_1(f_{n_0}) \\ \vdots & \ddots & \vdots \\ X_{n_0}(f_1) & ... & X_{n_0}(f_{n_0}) \end{matrix} \right\| = \|A\| \quad = U$$

where U is an unity of $\mathcal{O}_p$ and $\|A\|$ stands for the determinant of the matrix A. Without loss of generality, we assume that $X_i = \frac{\partial}{\partial x_i}$ for $i \leq n_0$ and $X_j(x_i) = 0$ for $i \leq n_0$ and $j > n_0$.

Thus, we diagonalize the matrix A in $\mathcal{O}_p$. But we need to be careful with such a diagonalization so to not destroy the R-monomial base.

- First Claim: Without loss of generality, $A(p)$ is a diagonal matrix. Indeed, notice that one just have to sum lines and columns of the matrix, which corresponds to a linear sum of the vector-fields and of the functions. At the end we get another set of vector-fields $\{\bar{X}_i\} = B[X_i]$ where B is a linear matrix. Taking the change of coordinates such that $\bar{X}_i = \frac{\partial}{\partial x_i}$ is equivalent to make a linear change of coordinates on the $(x_1, ..., x_{n_0})$ variables. As X_i for $i > n_0$ does not depend upon the first n_0 coordinates, we get that $\{\bar{X}_1, ..., \bar{X}_n\}$ is a R-monomial base;

- Second Claim: Without loss of generality $f_i = x_i$ for $i \leq n_0$. Indeed, take the change of coordinates $\bar{x}_1 = f_1$ and $\bar{x}_i = x_i$ otherwise. We remark that, after the change:

$$\bar{X}_1 = U\tfrac{\partial}{\partial \bar{x}_1}$$
$$\bar{X}_i = X_i + \alpha_i \tfrac{\partial}{\partial \bar{x}_1}$$

  for some unit U of $\mathcal{O}_p$. Notice that $\bar{X}_1$ is equivalent to $\frac{\partial}{\partial \bar{x}_1}$ and that $\{\bar{X}_1, \bar{X}_i - \frac{\alpha_i}{U}\bar{X}_1\}$ forms again a R-monomial base. So after performing this change of coordinates and reorganizing the base, we get another R-monomial base. Doing this change for all the other f_i we prove the claim.

Now we set $Y_i = X_i$ for $i \leq n_0$ and $Z_i = X_i$ for $i > n_0$. It is clear that $\{Y_i\}$ is totally transversal to $I_{\mathcal{C}}$ and that $\{Y_i, Z_j\}$ is a R-monomial base. Let us prove that $I_{\mathcal{C}}$ is invariant

by $\{Z_j\}$: As $\mathcal{I}_\mathcal{C}$ is θ-admissible, we conclude that $\Gamma_{n_0+1}(I_\mathcal{C}) \subset I_\mathcal{C}$. In particular, taking $Z = \sum h_j Z_j$ a $\mathcal{O}_p$-linear combination of the $\{Z_j\}$, we get:

$$\left\| \begin{array}{cccc} Y_1(f_1) & ... & Y_1(f_{n_0}) & Y_1(g) \\ \vdots & \ddots & \vdots & \vdots \\ Y_{n_0}(f_1) & ... & Y_{n_0}(f_{n_0}) & Y_{n_0}(g) \\ Z(f_1) & ... & Z(f_{n_0}) & Z(g) \end{array} \right\| \in I_\mathcal{C} \quad \Rightarrow \quad \left\| \begin{array}{cc} Id & Y_i(g) \\ 0 & Z(g) \end{array} \right\| \in I_\mathcal{C}$$

And, thus, $Z(g) \in I_\mathcal{C}$ for every $g \in I_\mathcal{C}$. Thus $I_\mathcal{C}$ is invariant by $\{Z_j\}$.

The only thing left to prove is that there exists a coordinate system such that $I_\mathcal{C} = (x_1, ..., x_t)$. For now, we have that $I_\mathcal{C} = (x_1, ..., x_{n_0}, h_1, ..., h_s)$ where h_i does not depend on the first n_0 variables. Consider the quotient $\mathcal{O}_p/(x_1, ..., x_{n_0})$. We get $\bar{I}_\mathcal{C} = (\bar{h}_i)$ and $\bar{\theta} = \{\bar{Z}_j\}$. Using proposition 3.2.1, there exists a change of coordinates such that $\bar{I}_\mathcal{C} = (\bar{x}_{n_0+1}, ..., \bar{x}_t)$ and $\{\bar{Z}_j\}$ is R-monomial. As neither Z_j nor h_i depends on $(x_1, ..., x_{n_0})$ the equivalent change of coordinates in $\mathcal{O}_p$ gives $I_\mathcal{C} = (x_1, ..., x_t)$. Moreover, $\{Y_i, Z_j\}$ is still R-monomial and $Z_j(x_i) = 0$ (for $i < n_0$). $\square$

Remark 3.3.2. *If a center $\mathcal{C}$ is θ-admissible, for each point $p \in \mathcal{C}$, we can decompose the distribution θ_p in two complementary parts such that: $I_\mathcal{C}$ is totally transversal to one part and invariant by the other.*

3.4 Proof of theorem 1.1.2

We present a proposition that trivially implies theorem 1.1.2:

Proposition 3.4.1. *Let (M, θ, E) be a d-foliated analytic manifold and $\sigma : (M', \theta', E') \to (M, \theta, E)$ a θ-admissible blow-up with center $\mathcal{C}$. Given $p \in \mathcal{C}$, $q \in \sigma^{-1}(p)$ and the coherent set of generators $< Y_i, Z_j >= \theta_p$ given by proposition 3.3.1:*

- $\theta'_q =< (\mathcal{O}(F) Y_i^*), \mathcal{O}_q.(Z_j^*) > .\mathcal{O}_q$;
- *If θ is R-monomial, so is θ'.*

Proof. By proposition 3.3.1, for each $p \in \mathcal{C}$ there exists a coordinate system $x = (x_1, ..., x_n)$ and a coherent set of generators $< Y_i, Z_j >= \theta_p$ such that $\mathcal{C}$ is totally transversal to $\{Y_i\}$ and invariant by $\{Z_j\}$ and $I_{\mathcal{C}} = (x_1, ..., x_t)$.

First, consider $X \in \theta_p$ a vector-field that leaves $\mathcal{C}$ invariant:

$$X = \sum A_i \frac{\partial}{\partial x_i}$$

The fact that $I_{\mathcal{C}}$ is invariant by X implies that $(A_i)_{i \leq t} \subset I_{\mathcal{C}}$. After the blow-up, without loss of generality, we assume that q is the origin of the x_1 chart:

$$(x_1, y_2, ..., y_t, x_{t+1}, ..., x_n) = (x_1, x_1 x_2, ..., x_1 x_t, x_{t+1}, ..., x_n)$$

In this chart, we get:

$$X^* = A_1^* \frac{\partial}{\partial x_1} + \sum_{i=2}^{t} \frac{1}{x_1}(A_i^* - A_1^* y_i) \frac{\partial}{\partial y_i} + \sum_{i=t+1}^{n} A_i^* \frac{\partial}{\partial x_i}$$

Moreover, as $(A_i)_{i \leq t} \subset I_{\mathcal{C}}$, we have that $\frac{1}{x_1} A_i^*$ is analytic. Thus, clearly X^* is analytic. In particular, this implies that Z_j^* are all analytic.

In the other hand, the expressions of the blow-up of the Y_i are given by:

- If $t = d$, we assume q is the origin of the x_1 chart and:

$$\begin{array}{c} Y_1^* = \frac{\partial}{\partial x_1}^* = \frac{1}{x_1}(x_1 \frac{\partial}{\partial x_1} - \sum_{i=t}^{m} y_i \frac{\partial}{\partial y_i}) \\ Y_i^* = \frac{\partial}{\partial x_i}^* = \frac{1}{x_1} \frac{\partial}{\partial y_i} \end{array} \tag{1}$$

- If $t > d$, then either q may be assumed as the origin of the x_1 chart or the origin of the x_t chart. In the first case the expressions are equal to 1. In the second, $Y_i^* = \frac{\partial}{\partial x_i}^* = \frac{1}{x_t} \frac{\partial}{\partial y_i}$ for all i.

Thus, they are all meromorphic and we must multiply by $\mathcal{O}(F)$ exactly one time to get analytic vector-fields. So, by lemma 2.3.2, $\theta_q^a =< \mathcal{O}(F).Y_i^*, Z_j^*, W_k > .\mathcal{O}_q$ where W_k is a combination of $Y_i^*.\mathcal{O}_q$ that is analytic and not contained in $< \mathcal{O}(F).Y_i^*, Z_j^* > .\mathcal{O}_q$. We have two cases to consider:

i If $d = t$, then there exists a linear combination that generates $W_1 = \frac{\partial}{\partial x_1}$. Thus: $\theta_q^a =< \mathcal{O}(F).Y_i^*, Z_j^*, W_1 = \frac{\partial}{\partial x_1} > .\mathcal{O}_q$;

ii If $d < t$, then it is clear by the expressions (1), there does not exist any W_k. This implies that $\theta_q^a =< \mathcal{O}(F).Y_i^*, Z_j^* > .\mathcal{O}_q$.

We now have to intersect θ^a with $Der_q(-logF)$. We claim that $\{\mathcal{O}(F).Y_i^*, Z_j^*\}.\mathcal{O}_q \subset Der_q(-logF)$. Indeed:

- It is clear by the expressions (1) that $\mathcal{O}(F)Y_i^*.\mathcal{O}_q$ leaves $F = \{x_1 = 0\}$ invariant.
- Consider $X \in \theta_p$ a vector-field that leaves $\mathcal{C}$ invariant. Then:

$$X^*(\mathcal{O}(F)) = X^*(\mathcal{I}_{\mathcal{C}}^*) = (X(\mathcal{I}_{\mathcal{C}}))^* \subset \mathcal{I}_{\mathcal{C}}^* = \mathcal{O}(F)$$

And, thus $Z_j^*.\mathcal{O}_q \subset Der_q(-logF)$.

So, clearly, in case $[ii]$, $\theta_q' = \theta_q^a =< (\mathcal{O}(F)Y_i^*), \mathcal{O}_q.(Z_j^*) > .\mathcal{O}_q$. In case $[i]$ we remark that W_1 does not leave F invariant. Moreover, $\mathcal{O}(F)Y_1^*.\mathcal{O}_q = x_1\frac{\partial}{\partial x_1}$ is the minimal multiple of W_1 that leaves F invariant. We conclude that $\theta_q' =< (\mathcal{O}(F)Y_i^*), \mathcal{O}_q.(Z_j^*) > .\mathcal{O}_q$.

Furthermore, if the $< Y_i, Z_k >$ is R-monomial:

$$Z_j = \sum_{i=1}^{n} \alpha_{i,j} x_i \frac{\partial}{\partial x_i}$$

with $\alpha_{i,j} \in R$. Without loss of generality, we assume that q is in the x_1-chart. We get:

$$Z_j^* = \sum_{i=j}^{t} (\alpha_{i,j} - \alpha_{1,j}) y_i \frac{\partial}{\partial y_1} + \sum_{i=t+1}^{n} \alpha_{i,j} x_i \frac{\partial}{\partial x_i}$$

that is clearly R-monomial at the origin. Moreover, using the expressions (1), it is clear that $< \mathcal{O}(F)Y_i^*, Z_j^* >$ is R-monomial at the origin. It rests to show that after the translation $(y_1, ..., y_n) = (x_1, ..., x_n) - q$, θ' is still R-monomial. We can further claim that:

Lemma 3.4.2. *The R-monomiality is an open condition.*

And the same proof of lemma 3.4.2 (see it below) is enough to show that θ' is also R-monomial at q. $\square$

Proof. (Lemma 3.4.2): Let θ be a R-monomial d-singular distribution over $p \in M$. There exists an open set $U \subset M$ containing p, a R-monomial coordinate system $x = (x_1, ..., x_n)$ defined over U and a R-monomial base $\{X_1, ..., X_d\}$ such that X_i is defined over U for all $i \leq d$. We claim that θ is R-monomial at every point $q \in U$.

Given $q \in U$, as $(x_1, ..., x_n)$ is a coordinate system over U, there exists $\xi \in \mathbb{K}^n$ such that $q = (\xi)$.

Let us first suppose that all X_i are singular at p:

$$X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$$

Without loss of generality, suppose that $\xi = (\xi_1, ..., \xi_t, 0, ..., 0)$ such that $\xi_i \neq 0$ for all $i \leq t$. Consider the matrix:

$$A = \begin{bmatrix} \alpha_{1,1} & ... & \alpha_{1,t} \\ \vdots & \ddots & \vdots \\ \alpha_{d,1} & ... & \alpha_{d,t} \end{bmatrix}$$

And let s be its rank. Apart from summing and multiplying the lines by elements of R:

$$A = \begin{bmatrix} D & B \\ 0 & 0 \end{bmatrix}$$

where D is a $s \times s$-diagonal matrix, B is a $s \times d - s$-matrix and both matrix have only elements in R. This implies that:

- $X_i = \alpha_{i,i} x_i \frac{\partial}{\partial x_i} + \sum_{j=s}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,i} \neq 0$ for all $i \leq s$;
- $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ for all $i > s$.

And all $\alpha_{i,j} \in R$. Now, taking the change of coordinates $(y_1, ..., y_n) = (x_1, ..., x_n) - \xi$ we obtain:

- $X_i = \alpha_{i,i}(y_i + \xi_i) \frac{\partial}{\partial y_i} + \sum_{j=s}^{t} \alpha_{i,j}(y_j + \xi_j) \frac{\partial}{\partial y_j} + \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}$ for all $i \leq s$;
- $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}$ for all $i > s$.

And $(0, ..., 0) = q$. We proceed with three coordinate changes:

- First change: let $y_i = \xi_i(-1 + \exp(\alpha_{i,i}\bar{y}_i))$ for all $i \leq s$ and $y_i = \bar{y}_i$ otherwise. One can easily check that this is a diffeomorphism in an open neighborhood of the origin and that:
$$\frac{\partial}{\partial \bar{y}_i} = \alpha_{i,i}(y_i + \xi_i)\frac{\partial}{\partial y_i}$$
for all $i < s$. This implies that:

  - $X_i = \frac{\partial}{\partial \bar{y}_i} + \sum_{j=s}^{t} \alpha_{i,j}(\bar{y}_j + \xi_j)\frac{\partial}{\partial \bar{y}_j} + \sum_{j=t+1}^{n} \alpha_{i,j}\bar{y}_j\frac{\partial}{\partial \bar{y}_j}$ for all $i \leq s$;
  - $X_i = \sum_{j=t+1}^{n} \alpha_{i,j}\bar{y}_j\frac{\partial}{\partial \bar{y}_j}$ for all $i > s$.

  In what follows, we drop the bars;

- Second change: let $y_i = -\xi_i + (\bar{y}_i + \xi_i)\exp(\sum_{j=1}^{s} \alpha_{j,i}\bar{y}_j)$ if $s < i \leq t$ and $\bar{y}_i = y_i$ otherwise. One can easily check that this is a diffeomorphism in an open neighborhood of the origin and that:
$$\frac{\partial}{\partial \bar{y}_i} = \frac{\partial}{\partial y_i} + \sum_{j=s}^{t} \alpha_{i,j}(y_j + \xi_j)\frac{\partial}{\partial y_j}$$
for all $i < s$. This implies that:

  - $X_i = \frac{\partial}{\partial \bar{y}_i} + \sum_{j=t+1}^{n} \alpha_{i,j}\bar{y}\frac{\partial}{\partial \bar{y}_j}$ for all $i \leq s$;
  - $X_i = \sum_{j=t+1}^{n} \alpha_{i,j}\bar{y}_j\frac{\partial}{\partial y_j}$ for all $i > s$.

  In what follows, we drop the bars;

- Third change: let $y_i = \bar{y}_i \exp(\sum_{j=1}^{s} \alpha_{j,i}\bar{y}_j)$ if $i > t$ and $\bar{y}_i = y_i$ otherwise. One can easily check that this is a diffeomorphism over the origin and that:
$$\frac{\partial}{\partial \bar{y}_i} = \frac{\partial}{\partial y_i} + \sum_{j=t+1}^{n} \alpha_{i,j}y_j\frac{\partial}{\partial y_j}$$
for all $i < s$ and
$$\bar{y}_i\frac{\partial}{\partial \bar{y}_i} = y_1\frac{\partial}{\partial y_i}$$
for $i > t$. This implies that:

– $X_i = \frac{\partial}{\partial \bar{y}_i}$ for all $i \leq s$;

– $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial y_j}$ for all $i > s$.

which forms a R-monomial base.

Now, we suppose that X_i for $i \leq r$ are non-singular at p. Without loss of generality, $X_i = \frac{\partial}{\partial x_i}$ and that $X_j(x_i) = 0$ whenever $i \leq r$ and $j > r$. In particular, when we make the translation $(y_1, ..., y_n) = (x_1, ..., n_n) - \xi$, $X_i = \frac{\partial}{\partial y_i}$ for $i \leq r$.

Consider the quotient $\mathcal{O}_U/(x_1, ..., x_r)$. It is another regular ring with a R-monomial singular distribution $\{\bar{X}_{r+1}, ..., \bar{X}_t\}$ that is all singular over the origin. Using the first part of the proof, there exists a change of coordinates in $\mathcal{O}_q$ $(x_1, ..., x_r)$ that turns $\{\bar{X}_{r+1}, ..., \bar{X}_t\}$ into a R-monomial base. Moreover, this coordinate change is invariant by the first r-coordinates. Taking the equivalent change in $\mathcal{O}_q$, we conclude the lemma. $\square$

4 Resolution theorem for an invariant ideal sheaf

4.1 The Hironaka's Theorem

Let us state the version of Hironaka's theorem that we are going to use:

Theorem 4.1.1. *(Hironaka): Let (M, E) be an analytic manifold with divisor, $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero) and $M_0 \subset M$ a relatively compact open set of M. Set $(M_0, E_0) = (M_0, E|_{M_0})$ and $\mathcal{I}_0 = \mathcal{I}.\mathcal{O}_{M_0}$. Then, there exists a resolution of $\mathcal{I}_0$:*

$$\mathcal{R}(M, M_0, \mathcal{I}, E) : (M_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_1} (M_0, E_0)$$

such that:

- *$\sigma : (M_r, E_r) \to (M_0, E_0)$ is an isomorphism over $M_0 \setminus V(\mathcal{I})$;*
- *$\mathcal{R}$ is functorial with respect to smooth morphisms.*

Remark 4.1.2. *The above theorem is an interpretation of theorem 1.3 of [2] or theorems 2.0.3 and 6.0.6 of [19] in the following sense:*

- *Theorem* 1.3 *of* [*2*] *is enunciated in algebraic category. But the paragraph before theorem* 1.1 *of* [*2*] *justifies the analytic statement;*

- *In* [*2*] *and* [*19*]*, the authors work with marked ideal sheafs. We avoid this by restricting the theorem for the marked ideal sheafs with weight one. The reader may verify that the definition of Support and (weak) transform give rise to the interpretations formulated above;*

- *In order to stress the functorial property of the resolution, we enunciate the result as presented in Kollar's book [8].*

We recall the notion of functoriality (see definition 3.31 of [8]). The functor $\mathcal{R}$ has:

- input: The category whose: objects are quadruples of the form $(M, M_0, \mathcal{I}, E)$; morphisms between $(M, M_0, \mathcal{I}, E_M)$ and $(N, N_0, \mathcal{J}, E_N)$ are smooth morphisms $\phi : M_0 \to N_0$ such that $\mathcal{J}_0.\mathcal{O}_{M_0} = \mathcal{I}_0$ and $\phi^{-1}(E_{N,0}) = E_{M,0}$;

- output: The category $\mathcal{B}l$ of blow-up sequences whose: objects are blow-up sequences:

$$\pi : (M_r, E_r) \to ... \to (M_0, E_0)$$

  with specified admissible centers $\mathcal{C}_i$; morphisms are given by the Cartesian product.

We remark that, for such a functor to be well defined, we accept blow-ups with empty centers (isomorphisms).

Remark 4.1.3. *The functoriality implies that: if* $\mathcal{C}_i$ *are the centers of* $\mathcal{R}(M, M_0, \mathcal{I}, E)$ *and* N *is an analytic manifold, then* $\mathcal{C}_i \times N$ *are the centers of* $\mathcal{R}(M\times N, M_0\times N, \mathcal{I}.\mathcal{O}_{M\times N}, E\times N)$.

An important consequence of the functoriality is the following global version of theorem 4.1.1:

Theorem 4.1.4. *Let* (M, E) *be an analytic manifold with divisor and* $\mathcal{I}$ *a coherent ideal sheaf (everywhere non-zero). Then there exists a proper analytic morphism:*

$$\mathcal{RG}(M, \mathcal{I}, E) = \sigma : \widetilde{M} \to M$$

such that:

- *for every $M_0 \subset M$ relatively compact open set, $\sigma|_{\sigma^{-1}M_0}$ is given by the finite sequence of blow-ups $\mathcal{R}(M, M_0, \mathcal{I}, E)$ of theorem* 4.1.1*;*

- *$\mathcal{RG}(M, \mathcal{I}, E)$ is functorial with respect to smooth morphisms.*

Where the functor $\mathcal{RG}(M, \mathcal{I}, E)$ has:

- Input: The category whose: objects are triples $(M, \mathcal{I}, E)$; morphisms between $(M, \mathcal{I}, E_M)$ and $(N, \mathcal{J}, E_N)$ are smooth morphisms $\phi : M \to N$ such that $\mathcal{J}.\mathcal{O}_M = \mathcal{I}$ and $\phi^{-1}(E_N) = E_M$;

- Output: The category $\mathcal{A}$ of proper analytic morphisms whose: objects are proper analytic morphisms $\sigma : \widetilde{M} \to M$; morphisms are given by Cartesian product.

The proof of theorem 4.1.4 follows the same steps of theorem 13.3 of [1]. We present the proof because the idea is useful for us, although we do not claim any originality.

Proof. (Theorem 4.1.4): Let $(U_i)_{i \in \mathbb{N}}$ be an open cover of M where each U_i is relatively compact and $U_i \subset U_{i+1}$. Then, for each U_i, take the morphism $\sigma_i : U_i^{'} \to U_i$ given by theorem 4.1.1. As $\epsilon_i : U_i \to U_{i+1}$ is a smooth morphism, there exists a morphism $\delta_i : U_i^{'} \to U_{i+1}^{'}$ such that the following diagram commutes:

$$\begin{array}{ccc} U_i^{'} & \xrightarrow{\delta_i} & U_{i+1}^{'} \\ \sigma_i \downarrow & & \downarrow \sigma_{i+1} \\ U_i & \xrightarrow{\epsilon_i} & U_{i+1} \end{array}$$

It is clear that M is isomorphic to the direct limit of the U_i i.e. the disjoint union $\cup U_i$ identified by the functions ϵ_i. Let $M^{'}$ be the direct limit of $U_i^{'}$ and $\sigma : M^{'} \to M$ the direct limit of σ. By construction, σ coincides with σ_i at each U_i. The functorial statement follows from the local functoriality of each σ_i. □

4.2 Statement of the result

Our intention, is to adapt the Hironaka's theorem in order to prove the following result:

Theorem 4.2.1. *Let (M, θ, E) be a d-foliated analytic manifold, $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero) and $M_0 \subset M$ a relatively compact open set of M. Set $(M_0, \theta_0, E_0) := (M_0, \theta|_{M_0}, E|_{M_0})$ and $\mathcal{I}_0 = \mathcal{I}.\mathcal{O}_{M_0}$. Suppose that $\mathcal{I}_0$ is invariant by θ_0 (i.e. $\theta_0[\mathcal{I}_0] \subset \mathcal{I}_0$). Then, there exists a resolution of $\mathcal{I}_0$:*

$$\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

such that:

- *Each $\mathcal{C}_i$ is invariant by θ_{i-1} (in particular, $\mathcal{C}_i$ is a θ_{i-1}-admissible center);*
- *$\sigma : (M_r, E_r) \to (M_0, E_0)$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$;*
- *If θ_0 is R-monomial, then so is θ_r;*
- *$\mathcal{R}_{inv}$ is functorial with respect to chain-preserving morphisms.*

Where the functor $\mathcal{R}_{inv}$ has:

- Input: The category whose: objects ate the quintuples $(M, M_0, \theta, \mathcal{I}, E)$; morphisms between $(M, M_0, \theta, \mathcal{I}, E_M)$ and $(N, N_0, \omega, \mathcal{J}, E_N)$ are smooth morphisms $\phi : M_0 \to N_0$ such that:
  - $\phi^{-1}(E_{N,0}) = E_{M,0}$;
  - $\mathcal{H}(\mathcal{I}, \theta, i).\mathcal{O}_{M_0} = \mathcal{H}(\mathcal{J}, \omega, i).\mathcal{O}_{M_0}$ for all $i \in \mathbb{N}$.

  Such morphisms are called *chain-preserving morphism*;
- Output: The category $\mathcal{B}l$ of blow-up sequences.

This functoriality property allows us to make a global statement just as before:

Theorem 4.2.2. *Let (M, θ, E) be a d-foliated analytic manifold and $\mathcal{I}$ a coherent ideal sheaf invariant by θ (everywhere non-zero). Then there exists a proper analytic morphism:*

$$\mathcal{RG}_{inv}(M, \theta, \mathcal{I}, E) = \sigma : (\widetilde{M}, \widetilde{\theta}) \to (M, \theta)$$

such that:

- *for every $M_0 \subset M$ a relatively compact open set of M, $\sigma|_{\sigma^{-1}M_0}$ is given by the finite sequence of blow-ups $\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}, E)$ of theorem* 4.2.1*;*

- *If θ is R-monomial, so is $\widetilde{\theta}$;*

- *$\mathcal{RG}_{inv}(M, \theta, \mathcal{I}, E)$ is functorial by chain-preserving morphisms.*

Where the functor $\mathcal{RG}_{inv}$ has:

- Input: The category whose: objects ate the quadraples $(M, \theta, \mathcal{I}, E)$; morphisms between $(M, \theta, \mathcal{I}, E_M)$ and $(N, \omega, \mathcal{J}, E_N)$ are chain preserving morphisms $\phi : M \to N$;

- Output: The category $\mathcal{A}$ of proper analytic morphism.

Proof. (Theorem 4.2.2): The proof follows, mutatis mutandis, the same proof of theorem 4.1.4. □

The main idea to prove theorem 4.2.1 is to consider the usual resolution $\mathcal{R}(M, M_0, \mathcal{I}, E)$ given by theorem 4.1.1 and to prove that it satisfies all the required properties.

4.3 Foliations - Geometrical invariance

According to the Stefan-Sussmann theorem (see [11] for an analytic version and [13, 12] for the more general theory) an involutive singular distribution θ is *integrable*. This means that, for all points $p \in M$, there exists an immersed locally closed sub manifold (N, i) passing through p such that:

- $D_q i(T_q N) = F_{i(q)}$ for all $q \in N$

where $F_q \subset TM_q$ is the linear subspace generated by θ_q. The maximal connected submanifolds with respect to this property are called *leafs*. The partition of M into leafs is called *singular foliation*.

A set S is said to be *geometrically invariant* by θ if every leaf that intersects S is totally contained in S. A coherent ideal sheaf $\mathcal{I}$ is said to be *geometrically invariant* by θ if $V(\mathcal{I})$ is geometrically invariant by θ. We need to relate this definition if the algebraic definition of invariance:

Lemma 4.3.1. *Let θ be an involutive d-singular distribution and $\mathcal{I}$ a coherent ideal sheaf. Then:*

- *I) If $\mathcal{I}$ is invariant by θ, it is geometrically invariant by θ;*
- *II) If $\mathcal{I}$ is geometrical invariant by θ and $\sqrt{\mathcal{I}} = \mathcal{I}$ then $\mathcal{I}$ is invariant by θ.*

Proof. We start supposing that θ is a 1-singular distribution. Take $p \in V(\mathcal{I}$ and let $\mathcal{F}$ be the leaf of θ through p.

- (I): If $\mathcal{F}$ is zero dimensional then it is clear that $\mathcal{F} \subset V(\mathcal{I}$, so we assume that $\mathcal{F}$ is unidimensional. In this case, for all $q \in \mathcal{F}$, θ_q is generated by a regular vector-field X_q and, by lemma 3.2.4, there exists a system of generators $\{f_1, ..., f_n\}$ of $\mathcal{I}.\mathcal{O}_q$ such that $X_q(f_i) \equiv 0$. This implies that $\mathcal{F} \cap V(\mathcal{I})$ is an open subset of $\mathcal{F}$. By analyticity it must also be closed and $\mathcal{F} \subset V(\mathcal{I})$.
- (II): We claim that $V(\theta[\mathcal{I}]) \supset V(\mathcal{I})$. This end the proof because:

$$\theta[\mathcal{I}] \subset \sqrt{\theta[\mathcal{I}]} \subset \sqrt{\mathcal{I}} = \mathcal{I}$$

  If $\mathcal{F}$ is zero-dimensional, then all vector-fields germs of θ_p are singular and it is clear that $p \in V(\theta[\mathcal{I}])$, so we assume that $\mathcal{F}$ is uni-dimensional. In this case, for all $q \in \mathcal{F}$, θ_q is generated by a regular vector-field X_q. For any $f \in \mathcal{I}.\mathcal{O}_p$, $f|_{\mathcal{F}} \equiv 0$, which implies that $X_p f|_{\mathcal{F}_p} \equiv 0$. As the choice of $f \in \mathcal{I}.\mathcal{O}_p$ is arbitrarily, $p \in V(\theta[\mathcal{I}])$.

Now, we prove the result for θ an involutive d-singular distribution. Take $p \in V(\mathcal{I}$ and let $\mathcal{F}$ be the leaf of θ through p and $\{X_1, ..., X_{d_p}\}$ be a set of coherent generators of θ in a small neighborhood U_p of p.

- (I): For a sufficiently small neighborhood U_p of p, every point $q \in U_p \cap \mathcal{F}_p$ is the image of the flux $(Fl_{t_1}^{X_1} \circ ... \circ Fl_{t_{d_p}}^{X_{d_p}})(p) = q$ for some $(t_1, ..., t_{d_p}) \in \mathbb{K}^z$, where $Fl_t^X(p)$ is the flux of the vector field X with time t and initial point p (see lemma 3.24 of [10]). As $X_i(\mathcal{I}.\mathcal{O}_{U_p}) \subset \mathcal{I}.\mathcal{O}_{U_p}$ by hypotheses, by the first part of the proof $U_p \cap \mathcal{F} \subset V(\mathcal{I})$. Thus, $V(\mathcal{I}) \cap \mathcal{F}$ is open in $\mathcal{F}$. As $V(\mathcal{I}) \cap \mathcal{F}$ is a closed subset of $\mathcal{F}$ by analyticity, $\mathcal{F} \subset V(\mathcal{I})$;

- (II): Take any $X \in \theta_p$ and let γ be the orbit of X at p. As $\mathcal{F} \subset V(\mathcal{I})$, it is clear that $\gamma \subset V(\mathcal{I})$ and, by the first part of the proof, $X(\mathcal{I}.\mathcal{O}_p) \subset \mathcal{I}.\mathcal{O}_p$. As the choice of the point and vector field is arbitrarily, we conclude that $\theta[\mathcal{I}] \subset \mathcal{I}$.

$\square$

Remark 4.3.2. *In particular, if $\mathcal{I}$ is a coherent regular ideal sheaf, it is invariant by θ if, and only if, it is geometrically invariant by θ.*

Remark 4.3.3. *Take $M = \mathbb{C}^2$, $\theta = \frac{\partial}{\partial x}$ and $\mathcal{I} = (xy, y^2)$. Then, $\mathcal{I}$ is not invariant by θ, but it is geometrically invariant by θ (notice that $\mathcal{I} \neq (y) = \sqrt{\mathcal{I}}$).*

4.4 Proof of theorem 4.2.1

In what follows, (M, θ, E) is a d-foliated analytic manifold, $M_0 \subset M$ is a relatively compact subset and $\mathcal{I}$ is a coherent ideal sheaf invariant by θ (everywhere non-zero):

Lemma 4.4.1. *Consider $\sigma : (M^{'}, \theta^{'}, E^{'}) \to (M, \theta, E)$ a θ-admissible blow-up with center $\mathcal{C}$ such that:*

- *$\mathcal{C} \subset V(\mathcal{I})$;*
- *$\mathcal{C}$ is geometrically invariant by θ.*

Then $\mathcal{I}^c$ is invariant by $\theta^{'}$.

Proof. As $\mathcal{C}$ is a regular sub-manifold geometrically invariant by θ, by lemma 4.3.1 it is also invariant by θ. This implies that $\theta^*[\mathcal{O}(F)] \subset \mathcal{O}(F)$. Moreover, proposition 3.4.1 implies that $\theta^{'} = \theta^*$. Thus:

$$\theta^{'}[\mathcal{I}^c] = \theta^*[\mathcal{I}^c] = \theta^*[\mathcal{I}^*.\mathcal{O}(-F)] \subset (\theta[\mathcal{I}])^*\mathcal{O}(-F) + \mathcal{I}^*\theta^*[\mathcal{O}(-F)] \subset \mathcal{I}^c$$

$\square$

We are ready to prove the Theorem 4.2.1:

Proof. (Theorem 4.2.1) By theorem 4.1.1, there exists a sequence of blow-ups:

$$\mathcal{R}(M, M_0, \mathcal{I}, E) = (M_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_1} (M_0, E_0)$$

that resolves $\mathcal{I}_0$. Define recursively $\theta_{i+1} = \theta_i^{'}$ and $\theta_0 = \theta.\mathcal{O}_{M_0}$. Now, suppose by induction that all centers $\mathcal{C}_i$ for $i < k$ are invariant by θ_{i-1}. We prove the result that $\mathcal{C}_k$ is invariant by θ_{k-1} (including $k = 1$).

First, notice that $\mathcal{I}^{c,k-1}$ is invariant by θ_{k-1} (by the recursive use of lemma 4.4.1). As $\mathcal{C}_k$ is regular, by lemma 4.3.1, we only need to prove that $\mathcal{C}_k$ is geometrically invariant by θ_{k-1}. We divide in two cases:

- First case: θ_{k-1} has leaf dimension one. Let $\mathcal{F}$ be a leaf of θ_{k-1} with non-empty intersection with $\mathcal{C}_k$ and take $p \in \mathcal{C}_k \cap \mathcal{F}$. If the leaf is zero-dimensional, clearly $\mathcal{F} \subset \mathcal{C}_k$, so assume that $\mathcal{F}$ is a one-dimensional leaf. Then, $\theta_{k-1,p}$ is generated by an unique non-singular vector-field X_p defined in an open neighborhood U_p of p. By the flow-box theorem there exists a coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ in U_p such that $X_p = \frac{\partial}{\partial x}$.

  By the coherence of $\mathcal{I}^{c,k-1}$ and proposition 3.2.2, $\mathcal{I}_{U_p} := \mathcal{I}^{c,k-1}.\mathcal{O}_{U_p}$ has a finite set of generators $\{f_1(y), ..., f_k(y)\}$ independent of x. Without loss of generality, $U_p = V \times W$ where V is a domain of $\mathbb{K}^{n-1}$ and W a domain of $\mathbb{K}$ such that:

  - The leafs of θ are given by $\{q\} \times W$, for every $q \in V$;
  - The divisor $E \cap U_p$ is equal to $E_V \times W$, where E_V is a SNC divisor over V.

  Furthermore, there exist a natural smooth morphism:

  $$\pi : V \times W \to V$$

  Define $g_i(y) = f_i(0, y)$ and let $\mathcal{J}$ be the ideal sheaf over $\mathcal{O}_V$ generated by the $(g_1(y), ..., g_t(y))$. As the f_i are invariant by x, we have that $\mathcal{J}.\mathcal{O}_{V\times W} = \mathcal{I}_{U_p}$ and, by the functorial property of theorem 4.1.1, the resolution of singularities of $\mathcal{J}$ and of $\mathcal{I}_{U_p}$ commutes. This implies that $\mathcal{C}_k = \pi(\mathcal{C}_k) \times W$ (see remark 4.1.3) and the intersection of $\mathcal{F} \cap \mathcal{C}_k$ must be open over $\mathcal{F}$. By analyticity it is also closed and $\mathcal{F} \subset \mathcal{C}_k$;

- Second case: θ_{k-1} has leaf dimensional d. Let $\mathcal{F}$ be a leaf of θ_{k-1} with non-empty intersection with $\mathcal{C}_k$ and take $p \in \mathcal{C}_k \cap \mathcal{F}$. Given a coherent set of generators $\{X_1, ..., X_{d_p}\}$ of $\theta_{k-1,p}$, there exists an open set U containing p where every point $q \in U \cap \mathcal{F}$ is in the image of the flux $Fl_{t_1}^{X_1} \circ ... \circ Fl_{t_{d_p}}^{X_{d_p}})(p)$ for some $(t_1, ..., t_{d_p})$. By another hand, if γ_i is the orbit of X_i passing through p, it is clear (by the first part of the proof) that $\gamma_i \subset \mathcal{C}_k$. This implies that $Fl_{t_1}^{X_1} \circ ... \circ Fl_{t_{d_p}}^{X_{d_p}})(p) \in \mathcal{C}_k$ for small enough $(t_1, ..., t_{d_p})$ and $\mathcal{F} \cap \mathcal{C}_k$ is open over $\mathcal{F}$. By analyticity it is also closed and $\mathcal{F} \subset \mathcal{C}_k$.

The functoriality is a direct consequence of the functoriality of theorem 4.1.1. The statement about R-monomiality is a direct consequence of theorem 1.1.2. $\square$

5 An ideal resolution subordinated to 1-foliations

5.1 Statement of the result

In this section, we present a result on resolution of ideal sheafs subordinated to a singular distributions with leaf dimension 1, without the hypotheses of invariance. When the leaf dimension is 1, the only non-trivial generalized fitting operation is Γ_1. Intuitively, this implies that the chain $\mathcal{C}h(\mathcal{I}, \theta)$ completely represents the interaction between $V(\mathcal{I})$ and θ and of their transforms under θ-admissible blow-ups of order one. The main result is the following:

Theorem 5.1.1. *Let (M, θ, E) be a 1-foliated analytic manifold, $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero) and $M_0 \subset M$ a relatively compact open set of M. Set $(M_0, \theta_0, E_0) := (M_0, \theta|_{M_0}, E|_{M_0})$ and $\mathcal{I}_0 = \mathcal{I}.\mathcal{O}_{M_0}$. Then, there exists a resolution of $\mathcal{I}_0$:*

$$\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, E_r) \overset{\sigma_r}{\to} ... \overset{\sigma_1}{\to} (M_0, \theta_0, E_0) = (M_0, \theta|_{M_0}, E|_{M_0})$$

such that:

- *Each $\mathcal{C}_i$ is θ_{i-1}-admissible;*
- *$\sigma : (M_r, \theta_r, E_r) \to (M_0, \theta_0, E_0)$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$;*
- *If θ_0 is R-monomial, then so is θ_r;*

- $\mathcal{R}_1$ *is functorial with respect to* 1*-chain-preserving morphisms.*

Where the functor $\mathcal{R}_1$ has:

- Input: The category whose: objects ate the quintuples $(M, M_0, \theta, \mathcal{I}, E)$ where θ has leaf dimension 1; morphisms are 1-*chain preserving morphisms*, i.e. chain preserving morphism between two quintuples whose singular distribution have leaf-dimension 1;
- Output: The category $\mathcal{B}l$ of blow-up sequences.

This functoriality property allows us to make a global statement just as before:

Theorem 5.1.2. *Let (M, θ, E) be a* 1*-foliated analytic manifold, and $\mathcal{I}$ a coherent ideal sheaf. Then there exists a proper analytic morphism:*

$$\mathcal{RG}_1(M, \theta, \mathcal{I}, E) = \sigma : (\widetilde{M}, \widetilde{\theta}) \to (M, \theta)$$

such that:

- *for every $M_0 \subset M$ relatively compact open set of M, $\sigma|_{\sigma^{-1}M_0}$ is given by the finite sequence of blow-ups $\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E)$ of theorem* 5.1.1*;*
- *If θ is R-monomial, so is $\widetilde{\theta}$;*
- $\mathcal{RG}_1(M, \theta, \mathcal{I}, E)$ *is functorial by* 1*-chain-preserving morphisms.*

Where the functor $\mathcal{RG}_1$ has:

- Input: The category whose: objects ate the quadraples $(M, \theta, \mathcal{I}, E)$; morphisms between $(M, \theta, \mathcal{I}, E_M)$ and $(N, \omega, \mathcal{J}, E_N)$ are 1-chain preserving morphisms $h : M \to N$;
- Output: The category $\mathcal{A}$ of proper analytic morphism.

Proof. (Theorem 5.1.2) The proof follows, mutatis mutandis, the same proof of theorem 4.1.4. $\square$

5.2 Sketch of the proof

Fix $M_0 \subset M$ an open relatively compact subset. The main invariant we consider is the pair:

$$(\nu, t) := (\nu_{M_0}(\mathcal{I}, \theta), type_{M_0}(\mathcal{I}, \theta))$$

The proof of the theorem shows that this invariant drops in two steps:

- First step: $(\nu, 2) \to (\nu, 1)$;
- Second step: $(\nu, 1) \to (\nu - 1, 2)$.

We need to make this steps functorial. The following propositions formalize the above idea:

Proposition 5.2.1. *Let (M, θ, E) be a d-foliated analytic manifold, $M_0 \subset M$ an open relatively compact subset and $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero). Set $(M_0, \theta_0, E_0) := (M_0, \theta|_{M_0}, E|_{M_0})$ and $\mathcal{I}_0 = \mathcal{I}.\mathcal{O}_{M_0}$ and suppose that $type_{M_0}(\mathcal{I}, \theta) = 2$. Then, there exists a sequence of θ-admissible blow-ups:*

$$\mathcal{S}1(M, M_0, \theta, \mathcal{I}, E) = (M_r, \theta_r, E_r) \overset{\sigma_r}{\to} ... \overset{\sigma_2}{\to} (M_1, \theta_1, E_1) \overset{\sigma_1}{\to} (M_0, \theta_0, E_0)$$

such that:

- $\nu_{M_r}(\mathcal{I}_0^{c,r}, \theta_r) \leq \nu_{M_0}(\mathcal{I}, \theta)$ *and* $type_{M_r}(\mathcal{I}^{c,r}, \theta_r) = 1$*;*
- *If $\phi : M_0 \to N_0$ is a chain-preserving morphism from $(M_0, \theta_0, \mathcal{I}_0, E_{M,0})$ to $(N_0, \omega_0, \mathcal{J}_0, E_{N,0})$, then there exists a chain-preserving morphism $\psi : M_r \to N_r$ from $(M_r, \theta_r, \mathcal{I}^{c,r}, E_{M,r})$ to $(N_r, \omega_r, \mathcal{J}^{c,r}, E_{N,r})$.*

Remark 5.2.2. *Proposition* 5.2.1 *is valid for any leaf dimension.*

Proposition 5.2.3. *Let (M, θ, E) be a 1-foliated analytic manifold, $M_0 \subset M$ an open relatively compact subset and $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero). Set $(M_0, \theta_0, E_0) := (M_0, \theta|_{M_0}, E|_{M_0})$ and $\mathcal{I}_0 = \mathcal{I}.\mathcal{O}_{M_0}$ and suppose that $type_{M_0}(\mathcal{I}, \theta) = 1$. Then, there exists a sequence of θ-admissible blow-ups:*

$$\mathcal{S}2(M, M_0, \theta, \mathcal{I}, E) = (M_r, \theta_r, E_r) \overset{\sigma_r}{\to} ... \overset{\sigma_2}{\to} (M_1, \theta_1, E_1) \overset{\sigma_1}{\to} (M_0, \theta_0, E_0)$$

such that:

- $\nu_{M_r}(\mathcal{I}^{c,r},\theta_r) < \nu_{M_0}(\mathcal{I},\theta)$*;*

- *If* $\phi : M_0 \to N_0$ *is a* 1*-chain-preserving morphism from* $(M_0, \theta_0, \mathcal{I}_0, E_{M,0})$ *to* $(N_0, \omega_0, \mathcal{J}_0, E_{N,0})$*, then there exists a* 1*-chain-preserving morphism* $\psi : M_r \to N_r$ *from* $(M_r, \theta_r, \mathcal{I}^{c,r}, E_{M,r})$ *to* $(N_r, \omega_r, \mathcal{J}^{c,r}, E_{N,r})$*.*

The main ideas for the proofs are:

- Proposition 5.2.1: let $\mathcal{I}_{\#}$ be the smallest invariant ideal sheaf containing $\mathcal{I}$. We claim that the resolution $\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}_{\#}, E)$, given by theorem 4.2.1, is the searched sequence of θ-admissible blow-ups over M_0;

- Proposition 5.2.3: let $\mathcal{I}_{M-tg} := H(\mathcal{I}, \theta, \nu - 1).\mathcal{O}_{M_0}$ be the maximal tangencies ideal sheaf of $\mathcal{I}$. We claim that the resolution $\mathcal{R}(M, M_0, \mathcal{I}_{M-tg}, E)$ given by theorem 4.1.1 is the searched sequence of θ-admissible blow-ups over M_0.

The details of the proofs relate blow-ups with the tangency chain of ideal sheafs $\mathcal{C}h(\mathcal{I}, \theta)$. The behaver of the chain under each of the proposition is different and will be studied separately.

But before proving the propositions, let us see how they prove theorem 5.1.1:

Proof. (Theorem 5.1.1): Take $N \subset M$ a relatively compact open subset such that $N \supset \overline{M}_0$ (where $\overline{S}$ stands for the topological closure of the set S). The order of tangency and type $(\nu(N), t(N)) := (\nu_M(\mathcal{I}, \theta), type_M(\mathcal{I}, \theta))$ are well-defined.

In particular $(\nu(N), t(N)) \geq (\nu, t) = (\nu_{M_0}(\mathcal{I}, \theta), type_{M_0}(\mathcal{I}, \theta))$ (where the order is lexicographically).

We claim that there exists $\overline{M}_0 \subset N_0 \subset N$ a relatively compact open subset and a sequence of θ-admissible blow-ups:

$$(N_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (N_0, \theta_0, E_0)$$

such that $(\nu(N_r), t(N_r)) < (\nu(N), t(N))$.

We prove the claim: Take any $\overline{M}_0 \subset N_0 \subset N$ relatively compact open subset and set. We assume that $(\nu(N_0), t(N_0)) = (\nu(N), t(N))$ (if $(\nu(N_0), t(N_0)) < (\nu(N), t(N))$ the claim is obvious). By propositions 5.2.1 or 5.2.3 applied to $(N, N_0, \theta, \mathcal{I}, E)$, there exists a sequence of θ-admissible blow-ups:

$$(N_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (N_0, \theta_0, E_0)$$

such that $(\nu(N_r), t(N_r)) < (\nu(N_0), t(N_0)) = (\nu(N), t(N))$, which proves the claim.

Now, it is clear that the recursive use of this claim proves the theorem: as the pair (ν, t) is inferiorly bordered by $(0, 1)$ one can not recursively apply the claim an infinite number of times. Once the process stops, we restrict all blow-ups to M_0 and its transforms (one can show that the restriction is well-defined because of the functoriality statements of propositions 5.2.1 and 5.2.3).

The functoriality statement of the theorem follows directly from the functoriality of propositions 5.2.1 and 5.2.3. Furthermore, as all blow-ups are θ-admissible, by theorem 1.1.2 if $\theta.\mathcal{O}_{M_0}$ is a R-monomial, so will be its transforms. $\square$

5.3 Proof of proposition 5.2.1

In what follows, (M, θ, E) is a d-foliated analytic manifold, $M_0 \subset M$ an open relatively compact subset, $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero) and $\nu = \nu_{M_0}(\mathcal{I}, \theta)$. We recall that $\mathcal{I}_{\#}$ stands for the smallest invariant ideal sheaf containing $\mathcal{I}$. Consider $\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}_{\#}, E)$ the sequence of θ-invariant blow-ups:

$$(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

given by theorem 4.2.1. We claim that this is the searched sequence of blow-ups for proposition 5.2.1. For simplifying notation, we won't mention the restriction to M_0, i.e. from now on $M = M_0$ (thus $\mathcal{I}_0 = \mathcal{I}$, $\theta_0 = \theta$ etc). This won't give us any problem because the only use

of $M \supset M_0$ is for well-applying theorem 4.2.1, which we have already done. We start with a lemma that relates the ideal chains with θ-admissible blow-ups by invariant centers:

Lemma 5.3.1. *Let $\sigma : (M', \theta', E') \to (M, \theta, E)$ be a θ-admissible blow-up by an invariant center $\mathcal{C} \subset V(\mathcal{I}_\#)$. Then:*

$$\mathcal{H}(\mathcal{I}^c, \theta', i) = \sigma^c(\mathcal{H}(\mathcal{I}, \theta, i)) = \mathcal{O}(-F)\sigma^*(\mathcal{H}(\mathcal{I}, \theta, i))$$

In particular: $(\mathcal{I}^c)_\# = (\mathcal{I}_\#)^c$.

Proof. As $\mathcal{C}$ is invariant by θ, by proposition 3.3.1, $\theta' = \theta^*$. Thus, for $\mathcal{J}$ a coherent ideal sheaf:

$$\theta'[\mathcal{O}(F)] \subset \mathcal{O}(F) \Rightarrow \mathcal{J}\theta'[(\mathcal{O}(-F))] \subset \mathcal{O}(-F)\mathcal{J}$$

In particular, this implies that:

$$\theta'[\mathcal{J}\mathcal{O}(-F)] + \mathcal{J}\mathcal{O}(-F) = \mathcal{O}(-F)(\theta'[\mathcal{J}] + \mathcal{J})$$

Notice that, as $\mathcal{I} \subset \mathcal{I}_\#$, $\mathcal{I}^c$ is well-defined and:

$$\mathcal{H}(\mathcal{I}^c, \theta', 0) = \mathcal{I}^c = \mathcal{O}(-F)\mathcal{I}^* = \mathcal{O}(-F)\sigma^*(\mathcal{H}(\mathcal{I}, \theta, 0))$$

We proceed by induction: suppose that $\mathcal{H}(\mathcal{I}^c, \theta', i) = \sigma^c(\mathcal{H}(\mathcal{I}, \theta, i))$ for $i < k$. Let us prove the result for $i = k$. It is clear that $H(\mathcal{I}, \theta, k) \subset \mathcal{I}_\#$, so:

$$\begin{aligned}
\mathcal{H}(\mathcal{I}^c, \theta', k) &= \mathcal{H}(\mathcal{I}^c, \theta', k-1) + \theta'[\mathcal{H}(\mathcal{I}^c, \theta', k-1)] = \\
&= \mathcal{H}(\mathcal{I}^c, \theta', k-1) + \theta^*[\mathcal{O}(-F)\mathcal{H}(\mathcal{I}, \theta, k-1)^*] = \\
&= \mathcal{O}(-F)(\mathcal{H}(\mathcal{I}, \theta, k-1) + \theta[\mathcal{H}(\mathcal{I}, \theta, k-1)])^* = \\
&= \mathcal{O}(-F)\mathcal{H}(\mathcal{I}, \theta, k)^* = \mathcal{H}(\mathcal{I}, \theta, k)^c
\end{aligned}$$

□

We are ready for proving the proposition:

Proof. (Proposition 5.2.1): In the notations of this section, we claim that:

$$(\mathcal{I}_\#)^{r,k} = (\mathcal{I}^{r,k})_\#$$

Indeed, notice that σ_1 is a blow-up by an invariant center $\mathcal{C}_1$ contained it $V(\mathcal{I}_{\#}) \subset V(\mathcal{I})$. By lemma 5.3.1 $\sigma_1^c(\mathcal{I}_{\#}) = (\sigma_1^c \mathcal{I})_{\#}$.

We now argue by induction: Suppose that for $i < k$: $\mathcal{C}_i \subset V(\mathcal{I}^{c,i-1})$ and that $(\mathcal{I}_{\#})^{c,i} = (\mathcal{I}^{c,i})_{\#}$. This implies that $\mathcal{C}_k \subset V((\mathcal{I}^{c,k-1})_{\#}) \subset V(\mathcal{I}^{c,k-1})$, and, by lemma 5.3.1 $(\mathcal{I}_{\#})^{c,k} = (\mathcal{I}^{c,k})_{\#}$, which proves the claim.

As $(\mathcal{I}_{\#})^{c,r} = \mathcal{O}_{M_r}$, $\mathcal{I}^{c,r}$ is of type 1 at M_r and $H(\mathcal{I}^{c,r}, \theta_r, \nu) = \mathcal{O}_{M_r}$, we further conclude that $\nu_{M_r}(\mathcal{I}^{c,r}, \theta_r) \leq \nu_{M_0}(\mathcal{I}, \theta) = \nu$.

For proving the functoriality statement, let $\phi : M_0 \to N_0$ be a chain preserving morphism between $(M, M_0, \theta, \mathcal{I}, E_M)$ and $(N, N_0, \omega, \mathcal{J}, E_N)$. Consider $\sigma : (M_r, \theta_r, E_{r,M}) \to (M_0, \theta_0, E_{M,0})$ and $\tau : (N_r, \omega_r, E_{N,r}) \to (N_0, \omega_0, E_{N,0})$ the sequence of blow-ups given above (the size of the sequence is the same by the functoriality statement of theorem 4.2.1). Theorem 4.2.1 guarantees the existence of a smooth morphism $\psi : M_r \to N_r$ such that the following diagram:

$$\begin{array}{ccc} M_r & \xrightarrow{\psi} & N_r \\ \sigma\downarrow & & \downarrow\tau \\ M_0 & \xrightarrow{\phi} & N_0 \end{array}$$

commutes. Given any ideal sheaf $\mathcal{K}$ over N_{i-1}, this implies that:

$$(\sigma_i)^*(\mathcal{K}.\mathcal{O}_{M_{i-1}}) = (\tau_i^* \mathcal{K}).\mathcal{O}_{M_i}$$

In particular, if $F_{i,M}$ is the exceptional divisor of $\sigma_i : M_i \to M_{i-1}$ and $F_{i,N}$ is the exceptional divisor of $\tau_i : N_i \to N_{i-1}$ (if τ_i is an isomorphism, $\mathcal{O}(F_{i,N}) = \mathcal{O}_{N_i}$) we have that:

$$\mathcal{O}(-F_{i,N}).\mathcal{O}_{M_i} = \mathcal{O}(-F_{i,M})$$

Suppose by induction that $\mathcal{H}(\mathcal{J}^{c,i}, \omega_i, j).\mathcal{O}_{M_i} = \mathcal{H}(\mathcal{I}^{c,i}, \theta_i, j)$ for $i < k$ and any $j \in \mathbb{N}$ (notice that for $k = 0$ the result is trivial). This implies that:

$$\begin{gathered} \mathcal{H}(\mathcal{J}^{c,k}, \omega_k, j).\mathcal{O}_{M_k} = (\mathcal{O}(-F_{k,N})\tau_k^* \mathcal{H}(\mathcal{J}^{c,k-1}, \omega_{k-1}, j)).\mathcal{O}_{M_k} = \\ = \mathcal{O}(-F_{k,M})\sigma_k^* \mathcal{H}(\mathcal{I}^{c,k-1}, \theta_{k-1}, j) = \mathcal{H}(\mathcal{I}^{c,k}, \theta_k, j) \end{gathered}$$

which ends the proof. $\square$

5.4 Proof of proposition 5.2.3

In what follows, (M, θ, E) is a 1-foliated analytic manifold, $M_0 \subset M$ an open relatively compact subset, $\mathcal{I}$ a coherent ideal sheaf (everywhere non-zero) and $\nu = \nu_{M_0}(\mathcal{I}, \theta)$. We call $\mathcal{I}_{M-tg} := \mathcal{H}(\mathcal{I}, \theta, \nu - 1)$ the maximal tangency ideal sheaf at M_0. By the hypotheses of proposition 5.2.3, this ideal sheaf is such that:

- $V(\mathcal{I}_{M-tg}) \cap M_0 \neq \emptyset$;
- $(\theta[\mathcal{I}_{M-tg}] + \mathcal{I}_{M-tg}).\mathcal{O}_{M_0} = \mathcal{O}_{M_0}$.

Consider $\mathcal{R}(M, M_0, \mathcal{I}_{M-tg}, E)$ the sequence of admissible blow-ups:

$$(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

given by theorem 4.1.1. We claim that this is the searched sequence of blow-ups for proposition 5.2.3. For simplifying notation, we won't mention the restriction to M_0, i.e. from now on $M = M_0$ (thus $\mathcal{I}_0 = \mathcal{I}$, $\theta_0 = \theta$ etc). This won't give us any problem because the only use of $M \supset M_0$ is for well-applying theorem 4.1.1, which we have already done. We start showing that the above resolution is θ-admissible:

Lemma 5.4.1. *Using the above notation, the sequence of blow-ups:*

$$(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} ... \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

is θ-admissible. Moreover, every center of blow-up $\mathcal{C}_i$ is totally transversal to θ_{i-1}.

Proof. For each $p \in V(\mathcal{I}_{M-tg})$ we claim that there exists a coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ such that $\{\frac{\partial}{\partial x}\}$ is a coherent set of generators of θ_p and $x \in \mathcal{I}_{M-tg}$.

Indeed, as the type of $\mathcal{I}$ at p is one, we conclude that θ_p is generated by a non-singular vector-field $\frac{\partial}{\partial x}$. Moreover, there exists $g \in \mathcal{I}_{M-tg}$ such that Xg is an unity. This implies that $g = xU(x, y) + h(y)$ where $U(x, y)$ is an unity. Making the change of coordinates $\bar{x} = g$ we get that $\bar{x} \in \mathcal{I}$ and $X = V\frac{\partial}{\partial \bar{x}}$, where $V = U(x, y) + xU_x(x, y)$ is an unity. Thus $\{\frac{\partial}{\partial \bar{x}}\}$ generates θ_p and the claim is proved.

In particular, this implies that $\mathcal{C}_1.\mathcal{O}_p \subset \{x = 0\}$ and $\mathcal{C}_1$ is totally transversal to θ_0. So, suppose by induction that, for $i < k$, $\mathcal{C}_i$ is totally transversal to θ_{i-1} and, for $p \in V(\mathcal{I}^{c,i}_{M-tg})$, there exists a coordinate system $(x, y_1, ..., y_{n-1})$ such that $\{\frac{\partial}{\partial x}\}$ is a coherent set of generators of $\theta_{i,p}$ and $x \in \mathcal{I}^{c,i}_{M-tg}$.

Take any $p \in \mathcal{C}_k$. As $\mathcal{C}_k \subset V(\mathcal{I}^{c,k-1}_{M-tg})$, by the induction hypotheses, there exists a coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ such that $\mathcal{C}_k.\mathcal{O}_p \subset \{x = 0\}$. This implies that $\mathcal{C}_k$ is totally transversal to θ_{k-1}. Now take $q \in V(\mathcal{I}^{c,k}_{M-tg})$. Either σ_k is a local diffeomorphism over q, and the result is obvious, or $q \in F_k$. At the later case, by the induction hypotheses there exists a coordinate system $(x, y_1, ..., y_{n-1})$ over $p = \sigma(q)$ such that $\frac{\partial}{\partial x}$ generates $\theta_{k-1,p}$ and $x \in \mathcal{I}^{c,k-1}_{M-tg}.\mathcal{O}_p$. Without loss of generality assume that q is the origin of the y_1-chart. It is easy to see that $\frac{\partial}{\partial x}$ generates $\theta_{k,q}$ (by proposition 3.3.1) and that $x \in \mathcal{I}^{c,k}_{M-tg}$. This ends the proof. □

Contrasting with the previous section, the prove of proposition 5.2.3 will need direct calculations over each step of the blow-up sequence:

Proof. (Proposition 5.2.3): The proof of this result needs explicit expressions of θ and the chain of ideals after blow-ups. By lemma 5.4.1 the center $\mathcal{C}_t$ is totally transversal to the foliation θ_{t-1}. So, by proposition 3.3.1 we get that:

$$\theta_{t+1} = \mathcal{O}(F_t)\sigma_t^*(\theta_t)$$

Using this expression recursively, we get:

$$\theta_t = \prod_{i=1}^{t}(i\sigma_t)^*(\mathcal{O}(F_i))(0\sigma_t)^*(\theta)$$

where we recall that $(i\sigma_t) = (\sigma_{i+1}, ..., \sigma_t)$. In particular, we have that:

$$\begin{gathered}\theta_t[(i\sigma_t)^*(\mathcal{O}(F_i))] = \textstyle\prod_{j=i}^t[(j\sigma_t)^*(\mathcal{O}(F_j))] * (i\sigma_t)^*(\theta_i)[(i\sigma_t)^*(\mathcal{O}(F_i))] = \\ = \textstyle\prod_{j=i}^t[(j\sigma_t)^*(\mathcal{O}(F_j))] * (i\sigma_t)^*[\theta_i\mathcal{O}(F_i)] = \textstyle\prod_{j=i}^t(j\sigma_t)^*(\mathcal{O}(F_j)) \subset \\ \subset (i\sigma_t)^*(\mathcal{O}(F_i))\end{gathered}$$

In particular, this implies that:

$$\theta_t[\textstyle\prod_{i=1}^t(i\sigma_t)^*\mathcal{O}(\alpha F_i)] \subset \textstyle\prod_{i=1}^t(i\sigma_t)^*\mathcal{O}(\alpha F_i)$$

for any $\alpha \in \mathbb{Z}$. For simplifying the notation, we define:

$$\mathcal{K}_t(\alpha) = \prod_{i=1}^{t} [(i\sigma_t)^* \mathcal{O}(\alpha F_i)]$$

So, we have proved that:

$$\theta_t = \mathcal{K}_t(1)(0\sigma_t)^*\theta \quad \text{and} \quad \theta_t(\mathcal{K}_t(\alpha)) \subset \mathcal{K}_t(\alpha) \tag{2}$$

Now, given any ideal sheaf $\mathcal{J}$, this equations implies that:

$$\theta_t[\mathcal{K}_t(\alpha)\mathcal{J}] + \mathcal{K}_t(\alpha)\mathcal{J} = \mathcal{K}_t(\alpha)(\mathcal{J} + \theta_t[\mathcal{J}]) \tag{3}$$

Now, let us calculate the transform of the chain of ideal sheafs at the level t. For obtaining these expressions we first assume that the blow-up sequence $\sigma = (\sigma_r, ..., \sigma_1)$ that resolves $\mathcal{I}_{M-tg}$ has centers $\mathcal{C}_i \subset V(\mathcal{I}^{c,i-1})$.

For simplifying the notation, we denote by $\tau_t := 0\sigma_t = (\sigma_1 \circ ... \circ \sigma_t)$ and $\mathcal{I}^* := \tau_t^*(\mathcal{I})$. We claim that the chain of ideals is given by the following expression:

$$\mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k) = \mathcal{K}_t(-1). \sum_{i=0}^{k} \mathcal{K}_t(i)\mathcal{H}(\mathcal{I}, \theta, i)^* \tag{4}$$

Let us prove the claim: It is clearly true for $k = 0$ or for $t = 0$. So, assume that it is already proved for $t - 1$ and for $0 \leq k < k_0$. Lets prove for $k = k_0$:

$$\begin{gathered}
\mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0) = \mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0 - 1) + \theta_t[\mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0 - 1)] = \\
= \mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0 - 1) + \theta_t[\mathcal{K}_t(-1). \textstyle\sum_{i=0}^{k_0-1} \mathcal{K}_t(i)\mathcal{H}(\mathcal{I}, \theta, i)^*] = \\
= \mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0 - 1) + \mathcal{K}_t(-1). \textstyle\sum_{i=0}^{k_0-1} \mathcal{K}_t(i)\theta_t[\mathcal{H}(\mathcal{I}, \theta, i)^*] = \\
= \mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0 - 1) + \mathcal{K}_t(-1). \textstyle\sum_{i=0}^{k_0-1} \mathcal{K}_t(i+1)\tau_t^*(\theta[\mathcal{H}(\mathcal{I}, \theta, i)]) = \\
= \mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0 - 1) + \mathcal{K}_t(-1). \textstyle\sum_{i=0}^{k_0-1} \mathcal{K}_t(i+1)\tau_t^*(\mathcal{H}(\mathcal{I}, \theta, i+1)) = \\
= \mathcal{K}_t(-1). \textstyle\sum_{i=0}^{k_0} \mathcal{K}_t(i)\mathcal{H}(\mathcal{I}, \theta, i)^*
\end{gathered}$$

(we have used equation 3 in the second line and 2 in the third). So the formula is proved. In particular, we have that:

$$\begin{gathered}
\mathcal{H}(\mathcal{I}^{c,t}, \theta_t, k_0) = \mathcal{K}_t(-1). \textstyle\sum_{i=0}^{k_0} \mathcal{K}_t(i)\mathcal{H}(\mathcal{I}, \theta, i)^* \subset \\
\subset \mathcal{K}_t(-1). \textstyle\sum_{i=0}^{k_0} \mathcal{H}(\mathcal{I}, \theta, i)^* = \sigma^c\mathcal{H}(\mathcal{I}, \theta, k_0)
\end{gathered}$$

This proves that:

$$\sigma^c \mathcal{H}(\mathcal{I}, \theta, \nu - 1) \supset \mathcal{H}(\mathcal{I}^c, \theta_t, \nu - 1) \tag{5}$$

In particular, this implies that $\mathcal{C}_i \subset V(\mathcal{I}^{c,i-1})$, as was assumed before. Indeed, for correctly arguing, we have to do it by induction: It is clear that $\mathcal{C}_1 \subset V(\mathcal{I})$. Assuming that $\mathcal{C}_i \subset V(\mathcal{I}^{c,i-1})$ for $i < t$, we obtain that the ideal chain at the $t-1$ level is given by the expression 4, thus 5 holds. In particular this implies that the $\mathcal{C}_t$ is contained in $V(\mathcal{H}(\mathcal{I}^{c,t-1}, \theta_{t-1}, \nu - 1)) \subset V(\mathcal{I}^{c,t-1})$.

Now, it rests to prove that after this sequence of blow-ups, the order of tangency decreases. In what follows, $\tau = (\sigma_1 \circ ... \circ \sigma_r)$ and we recall that $\mathcal{H}(\mathcal{I}, \theta, \nu - 1)^{c,r} = \mathcal{O}_{M_r}$.

$$\begin{gathered}
\mathcal{H}(\mathcal{I}^{c,r}, \theta_r, \nu - 1) = \mathcal{K}_r(-1). \textstyle\sum_{i=0}^{\nu-1} \mathcal{K}_r(i)\mathcal{H}(\mathcal{I}, \theta, i)^* = \\
= \mathcal{K}_r(-1). \textstyle\sum_{i=0}^{\nu-2} \mathcal{K}_r(i)\mathcal{H}(\mathcal{I}, \theta, i)^* + \mathcal{K}_r(\nu - 2) = \\
= \mathcal{H}(\mathcal{I}^{c,r}, \theta_r, \nu - 2) + \mathcal{K}_r(\nu - 2)
\end{gathered}$$

Thus:

$$\begin{gathered}
\theta_r[\mathcal{H}(\mathcal{I}^{c,r}, \theta_r, \nu - 1)] + \mathcal{H}(\mathcal{I}^{c,r}, \theta_r, \nu - 1) = \mathcal{H}(\mathcal{I}^{c,r}, \theta_r, \nu - 1) + \theta_r[\mathcal{K}_r(\nu - 2)] \subset \\
\subset \mathcal{H}(\mathcal{I}^{c,r}, \theta_r, \nu - 2) + \mathcal{K}_r(\nu - 2) = \mathcal{H}(\mathcal{I}^{c,r}, \theta_r, \nu - 1)
\end{gathered}$$

Which proves that the chain is stabilizing in at most $\nu - 1$ steps. So $\nu_{M_r}(\mathcal{I}^{c,r}, \theta_r) < \nu_{M_0}(\mathcal{I}, \theta)$.

For proving the functoriality statement, let $\phi : M_0 \to N_0$ be a chain preserving morphism from $(M, M_0, \theta, \mathcal{I}, E_M)$ to $(N, N_0, \omega, \mathcal{J}, E_N)$. Consider $\sigma : (M_r, \theta_r, E_{M,r}) \to (M_0, \theta_0, E_{M,0})$ and $\tau : (N_r, \omega_r, E_{N,r}) \to (N_0, \omega_0, E_{N,0})$ given above (the size of the sequence is the same by the functoriality statement of theorem 4.1.1). Theorem 4.1.1 guarantees the existence of a smooth morphism $\psi : M_r \to N_r$ such that the following diagram:

$$\begin{array}{ccc}
M_r & \xrightarrow{\psi} & N_r \\
\sigma\downarrow & & \downarrow\tau \\
M_0 & \xrightarrow{\phi} & N_0
\end{array}$$

commutes. Given any ideal sheaf $\mathcal{K}$ over N_{i-1}, this implies that:

$$(\sigma_i)^*(\mathcal{K}.\mathcal{O}_{M_{i-1}}) = (\tau_i^*\mathcal{K}).\mathcal{O}_{M_i}$$

In particular, if $F_{i,M}$ is the exceptional divisor of $\sigma_i : M_i \to M_{i-1}$ and $F_{i,N}$ is the exceptional divisor of $\tau_i : N_i \to N_{i-1}$ (if τ_i is an isomorphism, $\mathcal{O}(F_{i,N}) = \mathcal{O}_{N_i}$) we have that:

$$\mathcal{O}(-F_{i,N}).\mathcal{O}_{M_i} = \mathcal{O}(-F_{i,M})$$

Consider $\mathcal{K}_{t,M}(\alpha) = \prod_{i=1}^{t}[(i\sigma_t)^*\mathcal{O}(\alpha F_{i,M})]$ and $\mathcal{K}_{t,N}(\alpha) = \prod_{i=1}^{t}[(i\tau_t)^*\mathcal{O}(\alpha F_{i,N})]$. We have that:

$$\mathcal{K}_{i,N}(\alpha).\mathcal{O}_{M_i} = \mathcal{K}_{i,M}(\alpha)$$

This all implies that:

$$\mathcal{H}(\mathcal{J}^{c,r}, \omega_r, j).\mathcal{O}_{M_r} = (\mathcal{K}_{r,N}(-1) \sum_{i=0}^{j} \mathcal{K}_{r,N}(i)\mathcal{H}(\mathcal{J}, \omega, i)^*).\mathcal{O}_{M_k} =$$
$$\mathcal{K}_{r,M}(-1) \sum_{i=0}^{j} \mathcal{K}_{r,M}(i)\mathcal{H}(\mathcal{I}, \theta, i)^* = \mathcal{H}(\mathcal{I}^{c,r}, \theta_r, j)$$

which ends the proof. □

Acknowledgments

I would like to to express my gratitude to my advisor, professor Daniel Panazzolo, for the useful discussions, suggestions and revision of the manuscript. This work owns a great deal to his influence. I would also like to thank Orlando Villamayor and Santiago Encinas for the useful discussions on the subject.

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